THE DOCTRINE AND APPLICATION OF FLUXIONS.

CONTAINING
(Beſides what is common on the Subject)
A Number of NEW IMPROVEMENTS
in the THEORY.

AND

The SOLUTION of a Variety of New, and very
Intereſting, Problems in different Branches of
the MATHEMATICKS.

PART I. k

By THOMAS SIMPSON, F.R.S.

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MDCCLXXVI.

BRITISH MUSEUM

TO THE

RIGHT HONOURABLE

George Earl of *Macclesfield*.

MY LORD,

AS I eſteem it a very great Honour to be permitted to place the following Sheets under your Lordſhip's Protection, who are not only an Encourager of, but an Ornament to, Mathematical Learning; I have taken more than ordinary Pains, that, *What* is here uſhered into the World, with ſuch Advantage, may not be found altogether unworthy of ſo diſtinguiſhed a Patron.

I am not vain enough to imagine, that, to One ſo deeply read in *theſe* abſtruſe and curious Speculations, as your Lordſhip is univerſally

verſally allowed to be, this Work will appear without Faults: But then, I have the Satiſfaction to think, on the other hand, that, whatever is Here to be met with capable of bearing the Teſt of an exact and ſolid Judgment, will *alſo* have its due Weight, and not fail of receiving your Lordſhip's Approbation: And if, upon the Whole, there is Merit enough found to entitle me to a favourable Reception, it will gratify the higheſt Ambition of,

MY LORD,

Your LORDSHIP'S

Moſt Obedient Humble Servant,

Tho. Simpſon.

PREFACE.

HAVING, in the Year 1737, publiſhed a Piece, on this ſame Subject, under the Title of *A Treatiſe of Fluxions* (whereof the whole Impreſſion hath been long ſince ſold) it may be proper here, firſt of all, to aſſign the Reaſons why this Work is ſent abroad into the World as a New Book, rather than a Second Edition of the ſaid Treatiſe. Which, in ſhort, are theſe two: Firſt, becauſe the preſent Work is vaſtly more full and comprehenſive; and, ſecondly, becauſe the principal Matters in it which are alſo to be met with in that Treatiſe, are handled in a different Manner.

BESIDES the Preſs-Errors with which the ſaid Treatiſe abounds, there are ſeveral Obſcurities and Defects (which the Author's Want of Experience, and the many Diſadvantages he then laboured under, in his firſt Sally, may, it is hoped, in ſome meaſure excuſe.) But what is now offered to the Publick, being a Performance of more mature Conſideration and Judgment, it will, I flatter myſelf, be found much more correct, and claim a favourable Reception; eſpecially, as particular Care and Pains have been taken to put every Thing in a clear Light, and to oblige the lower, as well as the more experienced, Claſs of Readers.

THE Notion and Explication *Here* given of the firſt Principles of Fluxions, are not eſſentially different from what they are in the abovementioned Treatiſe, tho' expreſſed in other Terms. The Conſideration of Time, which I have introduced into the General Definition, will, perhaps, be diſliked by *Thoſe* who would have Fluxions to be *meer Velocities*: But the Advantage of conſidering them *otherwiſe* (not as the Velocities Themſelves, but the Magnitudes *They* would, uniformly, generate in a given finite Time) appear to me ſufficient to obviate any Objection on that Head.

BY taking Fluxions as *meer Velocities*, the Imagination is confined, as it were, to a Point, and, without proper Care, inſenſibly involved in metaphyſical Difficulties: But according to our Method of conceiving and explaining the Matter, leſs Caution in the Learner is neceſſary, and the higher Orders of Fluxions are rendered much more eaſy and intelligible —— Beſides, tho' Sir

Iſaac Newton defines Fluxions to be *the Velocities of Motions*, yet He hath Recourſe to the Increments, or Moments, generated in equal Particles of Time, in order to determine thoſe Velocities; which he afterwards teaches us to expound by finite Magnitudes of other Kinds: Without which (as is already hinted above) we could have but very obſcure Ideas of the higher Orders of Fluxions: For if Motion in (or at) a Point be ſo difficult to conceive, that, *Some* have, even, gone ſo far as to diſpute the very Exiſtence of Motion, how much more perplexing muſt it be to form a Conception, not only, of the Velocity of a Motion, but alſo in infinite Changes and Affections of *It*, in one and the ſame Point, where all the Orders of Fluxions are to be conſidered?

SEEING the Notion of a Fluxion, according to our Manner of defining It, ſuppoſes an uniform Motion, it may, perhaps, ſeem a Matter of Difficulty, at firſt View, how the Fluxions of Quantities, generated by Means of accelerated and retarded Motions, can be rightly aſſigned; ſince not any, the leaſt, Time can be taken during which the generating Celerity continues the ſame: Here, indeed, we cannot expreſs the Fluxion by any Increment or Space, *actually*, generated in a given Time (as in uniform Motions.) But, then, we can eaſily determine, what the contemporary Increment, or generated Space *would be*, if the Acceleration, or Retardation, was to ceaſe at the propoſed Poſition in which the Fluxion is to be found: Whence the true Fluxion, itſelf, will be obtained, without the Aſſiſtance of infinitely ſmall Quantities, or any metaphyſical Conſiderations.

THUS, for Example, the Motion of a Ball, deſcending by the Force of its own Gravity, is continually accelerated; but to have the Fluxion of the Diſtance fall'n thro' at any given Poſition of the Ball, we muſt find how far the Ball *would*, uniformly, deſcend, from that Point, in a given Time, if the Gravity, or the Earth's Attraction, from thence, was to ceaſe acting. By which Means we ſhall have as clear an Idea of the Fluxion and the true Meaſure of the Velocity of the Ball, at any Point aſſigned, as in thoſe Caſes where the Motion is, *actually*, uniform.

AGAIN, if a Right-line be ſuppoſed to move parallel to itſelf with an equable Motion, and to increaſe in Length, at the ſame Time; the Area generated thereby, will increaſe with an accelerated Velocity: But the Fluxion thereof, at any given Poſition of the Line, will be had by taking that Part of the Increment which *would*, uniformly, ariſe, was the Length (as well as the Velocity) of the Line to continue invariable from the propoſed Poſition. For, if the Length be ſuppoſed to increaſe, from the ſaid Poſition, the Area generated, from thence, will be, evidently, greater than That which would uniformly ariſe in the ſame Time; ſince the new Parts, produced each ſucceeding Moment, are greater and greater. Therefore the Fluxion muſt be leſs than any Space that can be deſcribed, in the given Time, when the Line increaſes. And, in the ſame Manner, the Fluxion will appear to be greater than any Space that can be deſcribed, in the ſame Time, when the Line decreaſes. *It* muſt, therefore, be equal to that Space, which will ariſe, when the Length of the generating Line, from the given Poſition, is ſuppoſed neither to increaſe nor decreaſe: Agreeable to *Art*. 4.

THUS much it ſeem'd proper to offer Here with regard to the Firſt Principles—I ſhall now proceed to ſay ſomething concerning the Order obſerv'd in treating, and putting together, the ſeveral Parts of the Work; wherein the Eaſe and Benefit of the younger Beginner have been particularly conſulted: To load ſuch an One with a Multitude of Rules and Precepts, before giving him any Taſte of their Uſe and Application, would, certainly, be very diſcouraging; and like obliging a Traveller to aſcend an high Mountain, without allowing him to ſtop by the Way, to take Breath, and refreſh his Spirits with a Proſpect of the agreeable and extenſive View he has to expect when he arrives at the Summit: I have therefore, after demonſtrating the Firſt Principles, proceeded immediately to exemplify their Utility in ſeveral entertaining Enquiries, before touching at all upon the Inverſe Method, or the more difficult Parts of the Direct. And, ſince that Branch of the Inverſe Method which treats of the Compariſon of Fluents is, naturally, ſomewhat difficult, it is referred to the Second Part of the Work, together with ſuch other Matters in the Theory as might appear, either, too tedious or hard to a Learner at firſt ſetting out. The like Care has been taken in the Diſpoſal of the reſt of the Work——As to the ſeveral Particulars whereof *It* is compoſed, I muſt refer to the Book itſelf, They being too many to be here enumerated: One Thing, however, I muſt not omit to take notice of, relating to that Part which treats of the aforeſaid Buſineſs of Fluents: To which it may, perhaps, be objected, That, notwithſtanding my having inſiſted ſo largely on the Subject, there are a Number of Forms of Fluxions and Fluents to be met with in Authors, that I have not ſo much as touch'd upon. This is granted; but then they are moſt of them ſuch as, I dare pronounce, can never ariſe in any Inquiry into Nature: And it would, doubtleſs, be Time and Labour miſapply'd, to ſwell the Work, and embarraſs the Learner with a Number of unneceſſary Difficulties, and empty Speculations; when what is, really, proper and uſeful, in the Subject, is ſufficient (it is well known) to exerciſe his utmoſt Attention and Reſolution.

I CANNOT put an End to this Preface without acknowledging my Obligations to a ſmall Tract, in-

intitled, *An Explanation of Fluxions in a Short Eſſay on the Theory*; printed for *W. Innys*: Wrote by a worthy Friend of mine (who was too modeſt to put his Name to that, his firſt, Attempt) whoſe Manner of determining the Fluxion of a Rectangle, and illuſtrating the higher Order of Fluxions, I have, in particular, follow'd, with little or no Variation.

The following **BOOKS** are all written by Mr. *Thomas Simpſon*, F. R. S. and printed for *J. Nourſe.*

1. THE Elements of Geometry; with their Application to the Menſuration of Superficies and Solids, to the Determination of the Maxima and Minima of Geometrical Quantities, and to the Conſtruction of a great Variety of Geometrical Problems, 8vo. the third Edition, 5s.

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THE
DOCTRINE and APPLICATION
OF
FLUXIONS.

PART the Firſt.

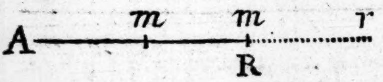
SECTION I.

Of the Nature, and Inveſtigation, of Fluxions.

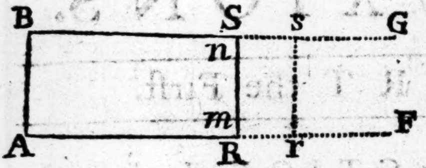
1. IN order to form a proper Idea of the Nature of Fluxions, all Kinds of Magnitudes are to be conſidered as generated by the *continual* Motion of ſome of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface.

2. Every Quantity ſo generated is called a variable, or flowing Quantity: *And the Magnitude by which any flowing Quantity* WOULD BE *uniformly increaſed in a given Portion of Time, with the generating Celerity at any propoſed Poſition, or Inſtant (was it from thence to continue invariable) is the Fluxion of the ſaid Quantity at that Poſition, or Inſtant.*

Thus, let the Point *m* be conceived to move from A, and generate the variable Right-line A*m*, by a Motion any how regulated; and let the Celerity thereof, when it arrives at any propoſed Poſition R, be ſuch *as would*, was it to continue uniform from that Point, be ſufficient to deſcribe the Diſtance, or Line R*r*, in the given Time allotted for the Fluxion: Then will R*r* be the Fluxion of the variable Line A*m*, in that Poſition.



3. The Fluxion of a plane Surface is conceived in like Manner, by ſuppoſing a given Right-line *mn* to move parallel to itſelf, in the Plane of the parallel, and immoveable Lines AF and BG: For, if (as above) Rr be taken to expreſs the Fluxion of the Line A*m*, and the Rectangle RrsS be completed; then that Rectangle, being the Space which *would be* uniformly deſcribed by the generating Line *mn*, in the Time that A*m* *would be* uniformly increaſed by *mr*, is therefore the Fluxion of the generated Rectangle B*m*, in that Poſition, according to the true Meaning of the Definition.



4. If the Length of the generating Line *mn* continually varies, the Fluxion of the Area will *ſtill* be expounded by a Rectangle under that Line and the Fluxion of the Abſciſſa, or Baſe: For let the curvilineal Space A*mn* be generated by the continual, and parallel, Motion of the (now) variable Line *mn*, and let Rr be the Fluxion of the Baſe, or Abſciſſa, *Am* (as before); then the Rectangle RrsS will, here alſo, be the Fluxion of the generated Space A*mn*: Becauſe, if the Length and Velocity of the generating Line *mn* were

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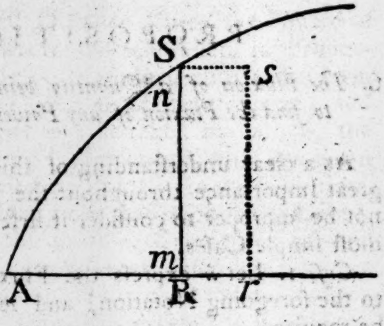


to continue invariable from the Poſition RS, the Rectangle R*rs*S would then be uniformly generated, with the very Celerity wherewith it begins to be generated, or with which the Space A*mn* is increaſed in that Poſition.

5. From what has been hitherto ſaid it will appear, that *the Fluxions of Quantities are, always, as the Celerities by which the quantities themſelves increaſe in Magnitude:* Whence it will not be difficult to form a Notion of the Fluxions of Quantities otherwiſe generated; as well ſuch as ariſe from the Revolution of Right-lines and Planes, as thoſe by parallel Motion: But of this hereafter. I come now to ſhew the Manner of determining the Fluxions of algebraic Quantities; by which all others, of what Kind ſoever, are explicable. But firſt of all it will be requiſite to premiſe the following Obſervations.

I. *That the final Letters* u, w, x, y, z *of the Alphabet are commonly put for variable Quantities; and the initial Letters* a, b, c, d, &c. *for invariable ones:* Thus the Diameter of a given Circle may be denoted by a, and the Sine of any Arch thereof (conſidered as variable) by x.

II. *That the Fluxion of a Quantity repreſented by a ſingle Letter, is uſually expreſſed by the ſame Letter with a Dot or Full-point over it:* Thus the Fluxion of x is repreſented by $\dot{x}$, and that of y by $\dot{y}$.

III. *That the Fluxion of a Quantity which decreaſes, inſtead of increaſing, is to be conſidered as negative.*

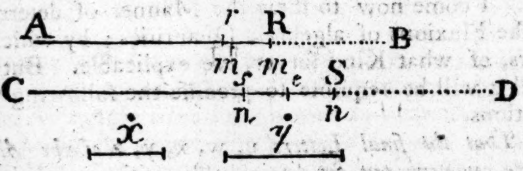
PROPOSITION I.

6. *The Fluxion of a Quantity being given, it is propoſed to find the Fluxion of any Power of that Quantity.*

As a clear underſtanding of this Problem will be of great Importance throughout the whole Work, it may not be improper to conſider it firſt in one or two of its moſt ſimple Caſes.

Caſe 1. Let $\dot{x}$ expreſs the Fluxion of x, (according to the foregoing Notation) and let the Fluxion of x^2 be required.

Conceive two Points m and n to proceed, at the ſame time, from two other Points A and C, along the Right-lines AB and CD, in ſuch ſort, that the Meaſure of the Diſtance CS (y), deſcribed by the latter, may be, *always*, equal to the Square of that AR (x), deſcribed by the former moving uniformly.



Furthermore, let r, s, and R, S, be any contemporary Poſitions of the generating Points, and let the Lines $\dot{x}$ and $\dot{y}$ repreſent the reſpective Diſtances that *would be* uniformly deſcribed, in the ſame time, with the Celerities of thoſe Points at R and S, then thoſe Lines will expreſs the Fluxions of Am and Cn in this Poſition, (*by the Definition*, Art. 2 and 5).

Moreover, ſince $Cs = Ar^2$ and $CS = AR^2$ (*by Hypotheſis*), if Rr be denoted by v, we ſhall have CS $(y) = x^2$, and $Cs\ (= \overline{x - v}|^2) = x^2 - 2xv + v^2$, and conſequently $Ss\ (= CS - Cs) = 2xv - v^2$; from whence we gather, that, while the Point m moves over the Diſtance v, the Point n moves over the Diſtance

$2xv-v^2$. But this laſt Diſtance (ſince the Square of any Quantity is known to increaſe faſter, in Proportion, than the Root) is not deſcribed with an uniform Motion (like the former), but an accelerated one; and therefore is equal to, and may be taken to expreſs, the uniform Space that might be deſcribed with the mean Celerity at ſome intermediate Point e, in the ſame time. Therefore, ſeeing the Diſtances that might be deſcribed, in equal times, with the uniform Celerity of m, and the mean Celerity at e, are to each other as v to $2xv-v^2$, or as 1 to $2x-v$, or, laſtly, as $\dot{x}$ to $2x\dot{x}-v\dot{x}$, (all which are in the ſame Proportion) it is evident, that, in the time the Point m would move uniformly over the Diſtance x, the other Point n, with its Celerity at e, *would* move uniformly over the Diſtance $2x\dot{x}-v\dot{x}$. This being the Caſe, let r, R, and s, S, be now ſuppoſed to coincide, by the Arrival of the generating Points at R and S, then e (being always between s and S, will likewiſe coincide with S; and the Diſtance, $2x\dot{x}-x\dot{x}$, which might be uniformly deſcribed in the aforeſaid time, with the Velocity at e, (now at S), will become barely equal to $2x\dot{x}$; which *(by the Defin.)* is equal to *($\dot{y}$)*, the true Fluxion of Cn or x^2 [a].

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[a] *It may, perhaps, ſeem inaccurate, that the Fluxions of x and x^2 are compared together, and expreſſed both by Lines, when the flowing Quantities themſelves, conſidered as a Right Line and a Square, admit of no Compariſon.——This Objection would, indeed, be of force, were the Expreſſions reſtrained to a geometrical Signification; but here our Notions are more abſtracted and univerſal, not obliging us to regard what Kind of Extenſion, may be defined by this or that Expreſſion, but only the Values of the algebraic Quantities thereby ſignified; to which the Meaſures of all other Quantities whatever are ultimately referred.——And, though Quantities of different Kinds cannot be compared with each other, their Meaſures, in Numbers, may.——Thus, for Inſtance, though it would be wrong to affirm, that a Square whoſe Area is 9 Inches is equal to a Line of 9 Inches long, yet it is no Impropriety at all to ſay the Numbers expreſſing their Meaſures, in Inches, are equal.*

7. *Caſe* 2. Let the Fluxion of x^3 be required.

Suppoſe every Thing to remain as in the preceding Caſe; only let C*n* be here equal to the Cube of A*m* (inſtead of the Square).

Then, in the very ſame manner, we have S*s* ($=CS - Cs = x^3 - \overline{x-v}|^3) = 3x^2v - 3xv^2 + v^3$: From whence it appears, that the Diſtances which *might be* deſcribed, in the ſame time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, in this Caſe, be to each other as v to $3x^2v - 3xv^2 + v^3$, or as $\dot{x}$ to $3x^2\dot{x} - 3xv\dot{x} + v^2\dot{x}$: Which laſt Expreſſion, when *s*, *e*, and S coincide (as before) will become $3x^2\dot{x}$, the true Fluxion of x^3 required.

8. *Univerſally*. Let C*n* be, *always*, equal to $\overline{Am}|^{n}$; alſo let $\overline{x-v}|^{n}$ (or $x-v$ raiſed to the Power whoſe Exponent is *n*) be repreſented by $x^n - ax^{n-1}v + bx^{n-2}v^2 - cx^{n-3}v^3$, &c. and let every Thing elſe be ſuppoſed as above.

Then, ſince S*s* ($x^n - \overline{x-v}|^{n}$) is $= ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c. it is plain that the Spaces which might be deſcribed, in the ſame time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, here, be to each other as v to $ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c. or as $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x} + cx^{n-3}v^2\dot{x}$, &c.

Therefore, all the Terms, wherein v is found, vaniſhing, when *s*, *e*, and S coincide, we have $ax^{n-1}\dot{x}$ for the required Fluxion of C*n*, or x^n; which Fluxion, becauſe the numeral Co-efficient of the ſecond Term of a Binomial involved is known to be, *univerſally*, equal to the Exponent of the Power, will alſo be truly expreſſed by $nx^{n-1}\dot{x}$. Q. E. I.

9. If the Quantity A*m* (or x) be generated with an accelerated, or a retarded Motion, inſtead of an uni-

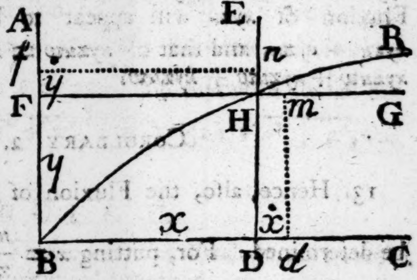
form one, the Fluxion of x^n (or C*n*) will come out exactly the ſame:

For the Spaces *r*R and *s*S, actually deſcribed in the ſame time, being always, to each other, in the Ratio of $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x}$, &c. the mean Celerities, at certain intermediate Points between *r*, R and *s*, S muſt, alſo, be in that Ratio: Which, when *v* vaniſhes (as above) will become that of $\dot{x}$ to $ax^{n-1}\dot{x}$, (or $nx^{n-1}\dot{x}$) the very ſame as before.

PROPOSITION II.

10. *To find the Fluxion of the Product or Rectangle of two variable Quantities.*

Conceive two Right-lines DE and FG, perpendicular to each other, to move, from two other Right-lines, BA and BC, continually parallel to themſelves, and thereby generate the Rectangle DF. Let the Path of their Interſection, or the Loci of the Angle H, be the Line BHR; alſo let D*d* ($\dot{x}$) and F*f* ($\dot{y}$) be the Fluxions of the Sides BD (x) and BF (y), and let *dm* and *fn*, parallel to DH and FH, be drawn. Therefore, becauſe the Fluxion of the Space or Area BDH is truly expreſſed by the Rectangle D*m* ($= y\dot{x}$ *) and that of the Area, or Space BFH, by the Rectangle F*n*, and equal Quantities have equal Fluxions, it follows that the Fluxion of the Rectangle $xy = \text{DF}$ ($= \text{BDH} + \text{BFH}$) is truly expreſſed by $yx + xy$. Q. E. I.



* Art. 4.

The ſame otherwiſe.

11. Let xy be the given Rectangle (as before); and put $z = x + y$, then z^2 being $= x^2 + 2xy + y^2$, we have $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 = xy$. But the Fluxion of $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, (and conſequently that of its Equal xy) is $z\dot{z} - x\dot{x} - y\dot{y}$ (by *Art.* 6): Which, becauſe $z = x + y$ and $\dot{z} = \dot{x} + \dot{y}$, is alſo equal to $\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$. Q. E. I.

COROLLARY 1.

12. Hence the Fluxion of the Product of three variable Quantities (yzu) may be derived: For, if x be put $= zu$; then yzu will become $= yx$, and its Fluxion $= y\dot{x} + x\dot{y}$ (*as above:*) But x being $= zu$, and, therefore, $\dot{x} = z\dot{u} + u\dot{z}$, if theſe Values be ſubſtituted in $y\dot{x} + \dot{x}y$, it will become $y \times \overline{z\dot{u} + u\dot{z}} + zu\dot{y} = yz\dot{u} + yu\dot{z} + zu\dot{y}$ the Fluxion of yzu required. In like Manner the Fluxion of $xyzu$ will appear to be $xyz\dot{u} + xy\dot{z}u + \dot{x}yzu + x\dot{y}zu$, and that of $xyzuw = xyzu\dot{w} + xyz\dot{u}w + xy\dot{z}uw + x\dot{y}zuw + \dot{x}yzuw$.

COROLLARY 2.

13. Hence, alſo, the Fluxion of a Fraction $\frac{u}{z}$ may be determined. For, putting $x = \frac{u}{z}$, we have $xz = u$, and therefore $x\dot{z} + z\dot{x} = \dot{u}$ (*as above*); whence, by Tranſpoſition and Diviſion, $\dot{x} = \frac{\dot{u}}{z} - \frac{x\dot{z}}{z} = \frac{\dot{u}}{z} - \frac{u\dot{z}}{z^2}$ (by writing $\frac{u}{z}$ for x) $= \frac{z\dot{u} - u\dot{z}}{z^2}$; which is the true Fluxion of x, or its Equal $\frac{u}{z}$, the Fraction propoſed.

14. Now, from the foregoing Propoſitions, and their ſubſequent Corollaries, the following practical Rules,

for determining the Fluxions of algebraic Quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a variable Quantity.

Multiply the Fluxion of the Root by the Exponent of the Power, and the Product by that Power of the ſame Root whoſe Exponent is leſs by Unity than the given Exponent.

This Rule is inveſtigated in Prop. 1, and is nothing more than $nx^{n-1}\dot{x}$ (the Fluxion of x^n) expreſſed in Words.

Hence the Fluxion of x^3 is $3x^2\dot{x}$; that of x^5 is $5x^4\dot{x}$; and that of $\overline{a+y}|^7$ is $7\dot{y}\times\overline{a+y}|^6$, (becauſe, a being conſtant, $\dot{y}$ is the true Fluxion of the Root $a+y$, in this Caſe).

Moreover the Fluxion of $\overline{a^2+z^2}|^{\frac{3}{2}}$, will be $\frac{3}{2}\times 2z\dot{z}\times\overline{a^2+z^2}|^{\frac{1}{2}}$, or $3z\dot{z}\sqrt{a^2+z^2}$: For here, x being put $= a^2+z^2$, we have $\dot{x} = 2z\dot{z}$, and therefore $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, the Fluxion of $x^{\frac{3}{2}}$ (or $\overline{a^2+z^2}|^{\frac{3}{2}}$) is $= 3z\dot{z}\sqrt{a^2+z^2}$, as above.

RULE II.

15. To find the Fluxion of the Product of ſeveral variable Quantities multiplied together.

Multiply the Fluxion of each, by the Product of the reſt of the Quantities, and the Sum of the Products thus ariſing will be the Fluxion ſought.*

*Art. 12.

Thus the Fluxion of xy, is $\dot{x}y + y\dot{x}$; that of xyz, is $xy\dot{z}+xz\dot{y}+yz\dot{x}$; and that of $xyzu$, is $xyz\dot{u}+xyu\dot{z}+xzu\dot{y}+yzu\dot{x}$.

RULE III.

16. To find the Fluxion of a Fraction.

From the Fluxion of the Numerator drawn into the Denominator, ſubtract the Fluxion of the Denominator drawn into the Numerator, and divide the Remainder by the Square of the Denominator *.

Thus, the Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{y^2}$; that of $\frac{x}{x+y}$, is $\frac{\dot{x}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times x}{\overline{x+y}|^{2}}=\frac{\dot{y}x-\dot{x}y}{\overline{x+y}|^{2}}$; and that of $\frac{x+y+z}{x+y}$, or $1+\frac{z}{x+y}$, is $\frac{\dot{z}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times z}{\overline{x+y}|^{2}}$; and ſo of others.

17. In the Examples hitherto given, each is reſolved by its own particular Rule; but in thoſe that follow, the Uſe of two, and ſometimes of all the three, Rules is requiſite.

Thus (*by Rule* 1. *and* 2.) the Fluxion of x^2y^2 is $2x^2y\dot{y}+2y^2x\dot{x}$; that of $\frac{x^2}{y^2}$ is $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$, (*by Rule* 1. *and* 3.) and that of $\frac{x^2y^2}{z}$ is $\frac{\overline{2x^2y\dot{y}+2y^2x\dot{x}}\times z-x^2y^2\dot{z}}{z^2}$; where all the three Rules are neceſſary.

When the propoſed Quantity is affected by a Co-efficient, or conſtant Multiplicator, the Fluxion found as above, muſt be multiplied by that Co-efficient or Multiplicator.

Thus, the Fluxion of $5x^3$ is $15x^2\dot{x}$. For, the Fluxion of x^3 being $3x^2\dot{x}$, that of $5x^3$, which is 5 times as great, muſt conſequently be $5\times3x^2\dot{x}$, or $15x^2\dot{x}$.

And, in the very ſame Manner the Fluxion of ax^n will appear to be $nax^{n-1}\dot{x}$. Moreover, the Fluxion of $\frac{a}{\overline{x^2+y^2}|^{\frac{1}{2}}}$, or $a\times\overline{x^2+y^2}|^{-\frac{1}{2}}$, will be expreſſed by

* Art. 13.

$a\times-\frac{1}{2}\times\overline{2x\dot{x}+2y\dot{y}}\times\overline{x^2+y^2}|^{-\frac{3}{2}}$, or $-\frac{a\times\overline{x\dot{x}+y\dot{y}}}{\overline{x^2+y^2}|^{\frac{3}{2}}}$;

that of $\sqrt{x+y^{\frac{1}{2}}}$, or $\overline{x+y^{\frac{1}{2}}}|^{\frac{1}{2}}$, by $\overline{\frac{1}{2}\dot{x}+\frac{1}{2}\times\frac{1}{2}\dot{y}y^{-\frac{1}{2}}}\times\overline{x+y^{\frac{1}{2}}}|^{-\frac{1}{2}}$, (*Rule* 1.) or $\frac{\frac{1}{2}\dot{x}+\frac{1}{4}\dot{y}y^{-\frac{1}{2}}}{\sqrt{x+y^{\frac{1}{2}}}}$, or $\frac{\frac{1}{2}\dot{x}y^{\frac{1}{2}}+\frac{1}{4}\dot{y}}{\sqrt{xy+y^{\frac{3}{2}}}}$;

and that of $\frac{\overline{x+a}|^2}{\sqrt{x^2-a^2}}$, or $\frac{\overline{x+a}|^2}{\overline{x^2-a^2}|^{\frac{1}{2}}}$, by

$\frac{2\dot{x}\times\overline{x+a}\times\overline{x^2-a^2}|^{\frac{1}{2}}-x\dot{x}\times\overline{x^2-a^2}|^{-\frac{1}{2}}\times\overline{x+a}|^2}{x^2-a^2}$; which

by Reduction, is $=\frac{2\dot{x}\times\overline{x^2-a^2}|^{\frac{1}{2}}-\dot{x}x\times\overline{x^2-a^2}|^{-\frac{1}{2}}\times\overline{x+a}}{x-a}$

$=\frac{2\dot{x}\times\overline{x^2-a^2}-x\dot{x}\times\overline{x+a}}{\overline{x-a}\times\overline{x^2-a^2}|^{\frac{1}{2}}}=\frac{2\dot{x}\times\overline{x-a}\times\overline{x+a}-\dot{x}x\times\overline{x+a}}{\overline{x-a}\times\sqrt{x^2-a^2}}$

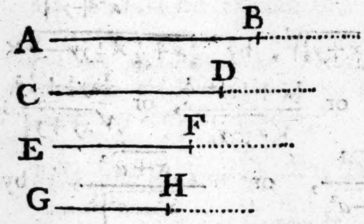
$=\frac{\overline{x+a}\times\overline{x\dot{x}-2a\dot{x}}}{\overline{x-a}\times\sqrt{x^2-a^2}}$.

Having explained the Manner of conſidering and determining the firſt Fluxions of variable or flowing Quantities, it will be proper to ſay ſomething, now, concerning the higher Orders, as Second, Third, Fourth, &c. Fluxions.

18. *The Second Fluxion of a Quantity is the Fluxion of the variable or algebraic Quantity expreſſing the Firſt Fluxion already defined *. By the Third Fluxion is meant the Fluxion of the variable Quantity expreſſing the Second: And by the Fourth, the Fluxion of the variable Quantity expreſſing the Third Fluxion: And ſo on.*

*Art. 2.

Thus, for Example, let the Line AB repreſent a variable Quantity, generated by the Motion of the Point B, and let the (firſt) Fluxion thereof (or the Space that *might be* uniformly deſcribed in a given Time, with the Celerity of B) be always expreſſed by the Diſtance of the Point D from a given, or fixed Point C: Then, if the Celerity of B be not every where the ſame; the Diſtance CD, expreſſing the Meaſure of that Celerity, muſt alſo vary, by the Motion of D, from, or towards C, according as the Celerity of B is an increaſing or a decreaſing one: And the Fluxion of the Line CD, ſo varying (or the Space (EF) that *might be* uniformly deſcribed in the aforeſaid given Time, with the Celerity of D) is the ſecond Fluxion of AB. Again, if the Motion of B be ſuch that neither it, nor that of D, (which depends upon it) be equable, then EF, expreſſing the Celerity of D, will alſo have its Fluxion GH; which is the third Fluxion of AB, and the ſecond Fluxion of CD.



And thus are the Fluxions of every other Order to be conſidered, *being the Meaſures of the Velocities by which their reſpective flowing Quantities, the Fluxions of the*
*Art. 2. *preceding Order, are generated* *.

19. Hence it appears, that a ſecond Fluxion always ſhews the rate of the Increaſe, or Decreaſe, of the firſt Fluxion; and that Third, Fourth, &c. Fluxions, differ in Nothing (except their Order and Notation) from Firſt Fluxions, being actually ſuch to the Quantities from whence they are immediately derived; and therefore are alſo determinable, in the very ſame Manner, by the general Rules already delivered.

Thus, *by Rule* 3. the (firſt) Fluxion of x^3 is $3x^2\dot{x}$: And, if $\dot{x}$ be ſuppoſed conſtant, that is, if the Root x be generated with an equable Celerity, the Fluxion of $3x^2\dot{x}$ (or $3\dot{x} \times x^2$) again taken, by the ſame Rule, will be $3\dot{x} \times 2x\dot{x}$, or $6x\dot{x}^2$; which therefore is the ſecond Fluxion of x^3: Whoſe Fluxion, found in like Sort, will be $6\dot{x}^3$, the third Fluxion of x^3. Farther than which we cannot go in this Caſe, becauſe the laſt Fluxion $6\dot{x}^3$ is here a conſtant Quantity.

20. In the preceding Example the Root x is ſuppoſed to be generated with an equable Celerity: But, if the Celerity be an increaſing or a decreaſing one, then $\dot{x}$, expreſſing the Meaſure thereof, being variable, will alſo have its Fluxion; which is uſually denoted by $\ddot{x}$: Whoſe Fluxion, according to the ſame Method of Notation, is again deſigned by $\dddot{x}$; and ſo on, with reſpect to the higher Orders.

21. Here follow a few Examples, wherein the Root x, (or y) is ſuppoſed to be generated with a variable Celerity.

Thus, the firſt Fluxion of x^3 is $3x^2\dot{x}$ (or $3x^2\times\dot{x}$). And, if the Fluxion of $3x^2\times\dot{x}$ (conſidered as a Rectangle) be, again, found *(by Rule* 2.) we ſhall have $6x\dot{x}\times\dot{x}+3x^2\times\ddot{x}=6x\dot{x}^2+3x^2\ddot{x}$, for the ſecond Fluxion of x^3.

Moreover, from the Fluxion laſt found we ſhall in like manner get $6\dot{x}\times\dot{x}^2+6x\times2\dot{x}\ddot{x}+6x\dot{x}\times\ddot{x}+3x^2\times\dddot{x}$ (or $6\dot{x}^3+18x\dot{x}\ddot{x}+3x^2\dddot{x}$) for the third Fluxion of x^3.

Thus alſo, if $\dot{y}=nx^{n-1}\dot{x}$, then will $\ddot{y}=n\times\overline{n-1}\times x^{n-2}\dot{x}^2+n\ddot{x}x^{n-1}$; and if $\dot{z}^2=\dot{x}\dot{y}$, then will $2\dot{z}\ddot{z}=\dot{x}\ddot{y}+\dot{y}\ddot{x}$: And ſo of others. But, in the Solution of Problems, it will be convenient to make the firſt Fluxion of ſome one of the ſimple Quantities (x or y) invariable, not only to avoid Trouble, but that it may ſerve as a Standard to which the variable Fluxions of the other Quantities, depending thereon, may be always referred. The Reader is alſo deſired here (once for all) to take particular Notice, that *the Fluxions of all Kinds and Orders, whatever, are contemporaneous, or ſuch as may be generated together, with their reſpective Celerities, in one and the ſame Time.*

SECT.

SECTION II.

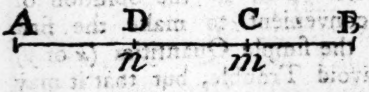
On the Application of Fluxions to the Solution of Problems DE MAXIMIS ET MINIMIS.

22. IF a Quantity, conceived to be generated by Motion, increaſes, or decreaſes, till it arrives at a certain Magnitude or Poſition, and then, on the contrary, grows leſſer or greater, and it be required to determine the ſaid Magnitude or Poſition, the Queſtion is called a Problem *de Maximis & Minimis.*

GENERAL ILLUSTRATION.

Let a Point *m* move uniformly in a Right Line, from A towards B, and let another Point *n* move after it, with a Velocity either increaſing, or decreaſing, but ſo that it may, at a certain Poſition, D, become equal to that of the former Point *m*, moving uniformly.

This being premiſed, let the Motion of *n* be firſt conſidered as an increaſing one; in which Caſe the Diſtance of *n* behind *m* will continually increaſe, till the two Points arrive at the cotemporary Poſitions C and D; but afterwards it will, again, decreaſe; for the Motion of *n*, till then, being ſlower than at D, it is alſo ſlower than that of the preceding Point *m* (by Hypotheſis) but becoming quicker, afterwards, than that of *m*, the Diſtance *mn* (as has been already ſaid) will again decreaſe: And therefore is a *Maximum*, or the greateſt of all, when the Celerities of the two Points are equal to each other.



But, if *n* arrives at D with a decreaſing Celerity; then its Motion being firſt ſwifter, and afterwards ſlower, than that of *m*, the Diſtance *mn* will firſt decreaſe and then increaſe; and therefore is a *Minimum*, or the leaſt of all, in the forementioned Circumſtance.

Since then the Diſtance *mn* is a *Maximum* or a *Minimum*, when the Velocities of *m* and *n* are equal, or when that Diſtance increaſes as faſt through the Motion of *m*, as it decreaſes by that of *n*, its Fluxion at that Inſtant is evidently equal to Nothing *. Therefore, as the Motion of the Points *m* and *n* may be conceived ſuch that their Diſtance *mn* may expreſs the Meaſure of any variable Quantity whatever, it follows, that the Fluxion of any variable Quantity whatever, when a Maximum or Minimum, is equal to Nothing.

* Art. 2 and 5.

EXAMPLE I.

23. *To divide a given Right-line* AB *into two ſuch Parts,* AC, BC, *that their Product, or Rectangle, may be the greateſt poſſible.*

Put the given Line AB $= a$, and let the Part AC, conſidered as variable (by the Motion of C from A towards B) be denoted by x: Then BC being $= a - x$, we have AC × BC $= ax - x^2$: Whoſe Fluxion $a\dot{x} - 2x\dot{x}$ being put $= 0$, according to the preſcript, we get $a\dot{x} = 2x\dot{x}$, and conſequently $x = \frac{1}{2}a$. Therefore AC and BC, in the required Circumſtance, are equal to each other: Which we alſo know from other Principles.

EXAMPLE II.

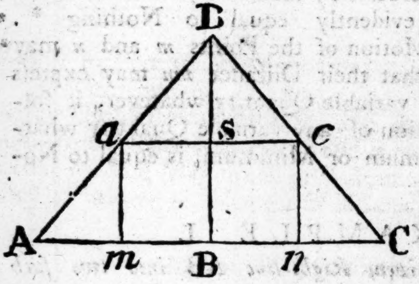
24. *To find the Fraction which ſhall exceed its Cube by the greateſt Quantity poſſible.*

Let x denote a variable Quantity, expreſſing Number in general; then the Exceſs of x above x^3 being univerſally repreſented by $x - x^3$, if the Fluxion thereof be taken, &c. we ſhall have $\dot{x} - 3x^2\dot{x} = 0$; and therefore $x = \sqrt{\frac{1}{3}}$, the Fraction required.

E X-

EXAMPLE III.

25. ***To determine the greateſt Rectangle that can be inſcribed in a given Triangle.***



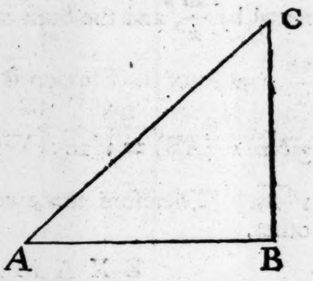
Put the Baſe AC of the given Triangle $= b$, and its Altitude BD $= a$; and let the Altitude (BS) of the inſcribed Rectangle *mc* (conſidered as variable) be denoted by x: Then, becauſe of the parallel Lines AC, and *ac*, it will be as BD (a) : AC (b) :: DS $(a-x)$: $\frac{ba-bx}{a} = ac$: Whence the Area of the Rectangle, or $ac \times$ BS will be $= \frac{bax-bx^2}{a}$: Whoſe Fluxion $\frac{ba\dot{x}-2bx\dot{x}}{a}$ being (as before) put $= 0$, we ſhall get $x = \frac{1}{2}a$. Whence the greateſt inſcribed Rectangle is that whoſe Altitude is juſt half the Altitude of the Triangle.

26. It will be proper to obſerve *here*, that the Value of a Quantity, when a *Maximum* or *Minimum*, may oftentimes be determined with more Facility by taking the Fluxion of ſome given Part, Multiple, or Power, thereof, than from the Fluxion of the Quantity itſelf. Thus, in the preceding Example, where the general Expreſſion is $\frac{bax-bx^2}{a} = \frac{b}{a} \times \overline{ax-x^2}$, if the conſtant Multiplicator $\frac{b}{a}$ be rejected, we ſhall have $ax-x^2$; whoſe Fluxion $a\dot{x}-2x\dot{x}$ being put $= 0$, we get $x = \frac{1}{2}a$, *the very ſame as before.*

The Reaſon of which is obvious; becauſe when the Quantity itſelf (be it of what Kind it will) is the greateſt, or leaſt poſſible, any *given* Part, Power, or Multiple of it is alſo the greateſt or leaſt poſſible.

EXAMPLE IV.

27. *Of all right-angled plain Triangles having the ſame given Hypothenuſe, to find that* (ABC) *whoſe Area is the greateſt.*



Let AC $=a$, AB$=x$, and BC $=y$: Then, x^2+y^2 being $=a^2$, we ſhall have $y=\sqrt{a^2-x^2}$, and conſequently $\frac{xy}{2}=\frac{x}{2}\sqrt{a^2-x^2}=$ the Area of the Triangle; whoſe Square $\frac{a^2x^2}{4}-\frac{x^4}{4}$ being, *alſo*, a *Maximum* *, *Art.26.
the Fluxion thereof $\frac{a^2x\dot{x}}{2}-x^3\dot{x}$ muſt therefore be $=0$, †: Whence x is found $=a\sqrt{\frac{1}{2}}$, and y †Art.22.
$(\sqrt{a^2-x^2})=a\sqrt{\frac{1}{2}}$.

The ſame otherwiſe.

Since $\frac{1}{2}xy$ is a *Maximum*, and $x^2+y^2=a^2$, let the Fluxions of both be taken, and you will have $\frac{1}{2}x\dot{y}+\frac{1}{2}y\dot{x}=0$, and $2x\dot{x}+2y\dot{y}=0$; from the former of which $\dot{y}$ will be $=-\frac{y\dot{x}}{x}$; and from the latter, it will be $=-\frac{x\dot{x}}{y}$: Therefore $\frac{y\dot{x}}{x}$ and $\frac{x\dot{x}}{y}$ are equal to each other, and conſequently $x=y$, (the ſame as before.)

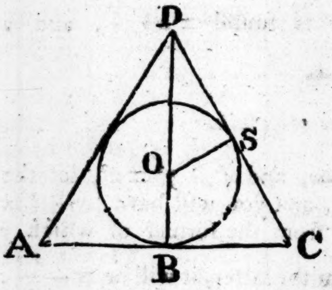
EXAMPLE V.

28. *Of all right-angled plain Triangles containing the ſame given Area, to find that whereof the Sum of the two Legs* AB+BC *is the leaſt poſſible.* (See the preceding Figure.)

Let one Leg, AB, be denoted by x, and the Area of the Triangle by a; then the other Leg will be denoted by $\frac{2a}{x}$, and the Sum of the two Legs will be $x+\frac{2a}{x}$; whereof the Fluxion is $\dot{x}-\frac{2a\dot{x}}{x^2}$; which, put $=0$, gives x (AB) $=\sqrt{2a}$: Whence BC $\left(\frac{2a}{x}\right)$ is alſo $=\sqrt{2a}$. Therefore the two Legs are equal to each other.

EXAMPLE VI.

29. *To determine the Dimenſions of the leaſt Iſoſceles Triangle* ACD *that can circumſcribe a given Circle.*



Let the Diſtance (OD) of the Vertex of the Triangle from the Center of the Circle, be denoted by x, and let the remaining Part of the Perpendicular, which is the Radius of the Circle, be repreſented by a: Then, if OS, perpendicular to DC, be drawn, we ſhall have DS $=\sqrt{x^2-a^2}$; and therefore, ſince DS : OS :: DB : BC, we likewiſe have BC $=\frac{a\times\overline{x+a}}{\sqrt{x^2-a^2}}$; which multiplied by $\overline{x+a}$ (BD) gives $\frac{a\times\overline{x+a}|^2}{\sqrt{x^2-a^2}}$ for the Area of the Triangle: Which being a *Minimum*, its Square muſt be a *Minimum*, and conſequently $\frac{\overline{x+a}|^4}{x^2-a^2}$, or its Equal $\frac{\overline{x+a}|^3}{x-a}$, a *Minimum* alſo *. Whoſe Fluxion, therefore, which is $\frac{3\dot{x}\times\overline{x+a}|^2\times\overline{x-a}-\dot{x}\times\overline{x+a}|^3}{\overline{x-a}|^2}$, being put $=0$, and the Whole divided by $\frac{\dot{x}\times\overline{x+a}|^2}{\overline{x-a}|^2}$, we alſo get $3\times\overline{x-a}-\overline{x+a}=0$; whence $x=2a$: Therefore, OD being $=2$OS, and the Triangles ODS and BDC equiangular, it is evident that DC is likewiſe $=2$BC $=$ AC; and ſo the Triangle ACD, when the leaſt poſſible, is equilateral.

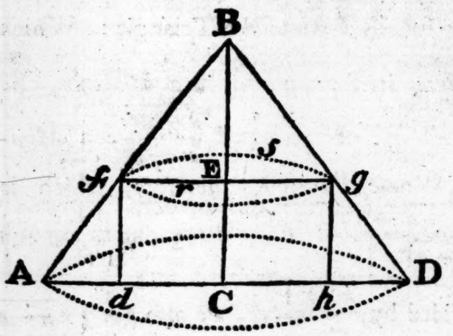
*Art. 26.

EXAMPLE VII.

30. *To determine the greateſt Cylinder, dg, that can be inſcribed in a given Cone* ADB.

Let $a=$BC, the Altitude of the Cone;
$b=$AD, the Diameter of its Baſe;
$x=fg\ (dh)$ the Diameter of the Cylinder, conſidered as variable;
$p=\left(\frac{3{,}14159,\ \&c.}{4}\right)$ the Area of the Circle whoſe Diameter is Unity.

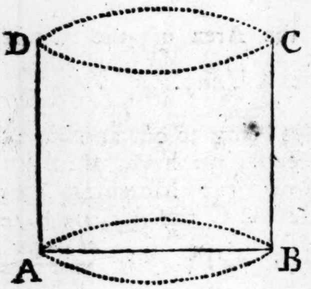
Then, the Areas of Circles being to one another as the Squares of their Diameters, we have, $1^2 : x^2 :: p : (px^2)$ the Area of the Circle *fsgr*: Moreover, from the Similarity of the Triangles ABC and A*df*, we have $\frac{1}{2}b$ (AC) : a (BC) :: $\frac{1}{2}b-\frac{1}{2}x$ (A*d*) : $df=\frac{ab-ax}{b}$; which multiplied by the Area px^2 (found above) gives



$\frac{pabx^2 - pax^3}{b}$ for the ſolid Content of the Cylinder: Which being a *Maximum*, its Fluxion $\frac{2pabx\dot{x}}{b} - \frac{3pax^2\dot{x}}{b}$ muſt be $= 0$ *, conſequently $x = \frac{2b}{3}$ and $df = \frac{a}{3}$: From whence it appears, that the inſcribed Cylinder will be the greateſt poſſible, when the Altitude thereof is juſt $\frac{1}{3}$ of the Altitude of the whole Cone.

EXAMPLE VIII.

31. *To determine the Dimenſions of a cylindric Meaſure* ABCD, *open at the Top, which ſhall contain a given Quantity (of Liquor, Grain,* &c.) *under the leaſt internal Superficies poſſible.*



Let the Diameter $AB = x$, and the Altitude $AD = y$; moreover let p (3,14159, *&c.*) denote the Periphery of the Circle whoſe Diameter is Unity, and let c be the given Content of the Cylinder. Then it will be $1 : p :: x : (px)$ the Circumference of the Baſe; which, multiplied by

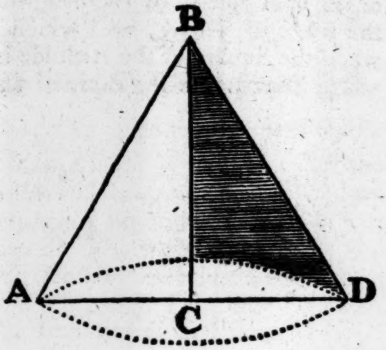
* Art. 22.

by the Altitude y, gives pxy for the concave Superficies of the Cylinder. In like Manner, the Area of the Baſe, by multiplying the ſame Expreſſion into $\frac{1}{4}$ of the Diameter x, will be found $= \frac{px^2}{4}$; which drawn into the Altitude y, gives $\frac{px^2y}{4}$ for the ſolid Content of the Cylinder; which being made $= c$, the concave Surface pxy will be found $= \frac{4c}{x}$, and conſequently the whole Surface $= \frac{4c}{x} + \frac{px^2}{4}$: Whereof the Fluxion, which is, $-\frac{4c\dot{x}}{x^2} + \frac{px\dot{x}}{2}$, being put $= 0$, we ſhall get $-8c + px^3 = 0$; and therefore $x = 2\sqrt[3]{\frac{c}{p}}$: Further, becauſe $px^3 = 8c$, and $px^2y = 4c$, it follows, that $x = 2y$; whence y is alſo known, and from which it appears, that the Diameter of the Baſe muſt be juſt the Double of the Altitude.

EXAMPLE IX.

32. *Of all Cones under the ſame given Superficies* (s) *to find that* (ABD) *whoſe Solidity is the greateſt.*

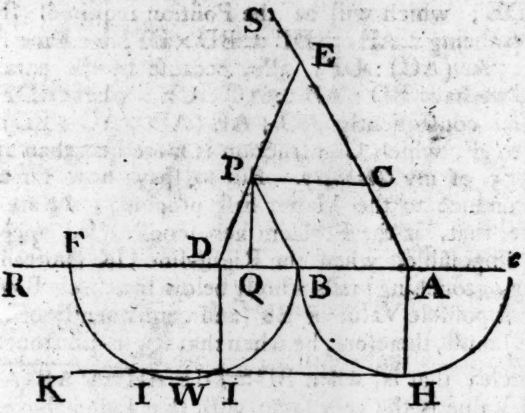
Let the Semidiameter of the Baſe, AC $= x$, and the Length of the ſlant Side AB $= y$; and let p (as in the preceding Examples) denote the Periphery of the Circle whoſe Diameter is Unity.



Then the Circumference of the Baſe will be $=2px$, the Area of the Baſe $=px^2$, and the convex Superficies of the Cone $=pxy$, (which laſt is found by multiplying half the Periphery of the Baſe by the Length of the ſlant Side): Wherefore, ſince the whole Superficies is $=px^2+pxy=s$, we have $y=\frac{s}{px}-x$; whence the Altitude CB $(\sqrt{AB^2-AC^2})=\sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$; which multiplied by $\left(\frac{px^2}{3}\right)$ $\frac{1}{3}$ of the Area of the Baſe, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$ for the ſolid Content of the Cone. Which being a *Maximum*, its Square $\frac{s^2x^2}{9}-\frac{2psx^4}{9}$ muſt alſo be a *Maximum*; and therefore $\frac{2s^2x\dot{x}}{9}-\frac{8psx^3\dot{x}}{9}=0$; whence $s-4px^2=0$, and conſequently $x=\sqrt{\frac{s}{4p}}$: From which $y\left(=\frac{s}{px}-x=\frac{s-px^2}{px}=\frac{3px^2}{px}=3x\right)$ will likewiſe be known; and from whence it will appear that the greateſt Cone under a given Surface, (or a given Cone under the leaſt Surface) will be when the Length of the ſlant Side is to the Semi-diameter of the Baſe in the Ratio of 3 to 1, or, (which comes to the ſame) when the Square of the Altitude is to the Square of the whole Diameter in the Ratio of 2 to 1.

EXAMPLE X.

33. *To determine the Poſition of a Right-line* DE, *which, paſſing through a given Point* P, *ſhall cut two Right-lines* AR *and* AS, *given by Poſition, in ſuch ſort that the Sum of the Segments,* AD *and* AE, *made thereby, may be the leaſt poſſible.*



Make PB, parallel to AS, $= a$, and PC, parallel to AR, $= b$; and let BD $= x$: Then, by reaſon of the parallel Lines, it will be, $x : a :: b : \text{CE} = \frac{ab}{x}$: Therefore $\text{AD} + \text{AE} = b + x + a + \frac{ab}{x}$, and its Fluxion, $\dot{x} - \frac{ab\dot{x}}{x^2}$, which, in the required Circumſtance, being $= 0$, we have $x^2 - ab$ alſo $= 0$, and conſequently $x = \sqrt{ab}$; whence the Poſition of DE is known. But the ſame Thing may be otherwiſe determined, independent of Fluxions, from the general Solution of the Problem for finding the Poſition of DE, when the Sum of the Segments AD and AE (inſtead of being a *Minimum*) ſhall be equal to a given Quantity. Of which Problem, the geometrical Conſtruction may be as follows.

Compleat the Parallelogram ABPC (as before) and, in RA produced, take Ac=AC, and let cF be equal to the given Sum of the two Segments: Alſo let two Semi-circles be deſcribed upon Bc and BF, and let AH, perpendicular to Bc, interſect the former in H; likewiſe let HK, parallel to Fc, interſect the latter in I; draw ID perpendicular to Fc, and, through P and D draw DE; which will be the Poſition required. For $AB\times Ac$ being $=AH^2=DI^2=BD\times DF$, we have BD : AB :: Ac (AC) : DF; alſo, becauſe of the parallel Lines, we have BD : AB :: AC : CE; whence DF=CE, and conſequently AD+AE (AD+AC+FD) is equal to cF, which Conſtruction is more neat than that in *p.* 155. of my *Geometry.* But to ſhew how far this may conduce to the Matter firſt propoſed; we are to obſerve, that, as the Problem here conſtructed appears to be impoſſible, when the Right-line HK (inſtead of cutting or touching) falls wholly below the Circle BWF, the leaſt poſſible Value of BF (and conſequently of AD + AE) muſt, therefore, be when that Right-line touches the Circle; that is, when $BD=DI=AH=\sqrt{AB\times AC}$; which Value is the very ſame with that found above.

The ſame Concluſion may alſo be deduced from the algebraic Solution of the foreſaid Problem: For, putting $b+x+a+\frac{ab}{x}$ (AD + AE) $=s$, and ſolving the Equation, x will be found $=\frac{s-a-b}{2}\pm\sqrt{\frac{\overline{s-a-b}|^2}{4}-ab}$: Which Equation being no longer poſſible than till $\frac{\overline{s-a-b}|^2}{4}-ab$ is $=0$, we have x, in that Circumſtance, $=\frac{s-a-b}{2}=\sqrt{ab}$; *ſtill as before.* In like Manner the *Maxima* and *Minima* may be determined in other Caſes, by finding the Poſition or Circumſtance wherein the general Problem begins to be impoſſible, (ſuppoſing the Quantity ſought to be given). But the Operation by

Fluxions is, for the general Part, much more ſhort and expeditious.

EXAMPLE XI.

34. *The ſame being given as in the preceding Example, to determine the Poſition, when the Line* DE, *itſelf, is the leaſt poſſible.*

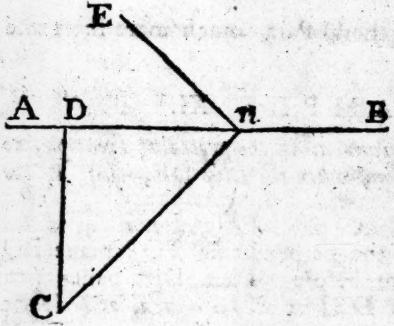
Upon AF let fall the perpendicular PQ; make BQ $=c$, and, the reſt, as before: Then DP^2 being ($=$ DB$^2+$BP$^2-2$BQ $\times$ DB) $= x^2+a^2-2cx$, and DB^2 : DP^2 :: DA^2 : DE^2, we have $x^2 : x^2+a^2-2cx :: \overline{b+x}|^2$: DE$^2 = \dfrac{\overline{b+x}|^2 \times \overline{x^2-2cx+a^2}}{x^2} = \overline{b+x}|^2 \times \overline{1-\dfrac{2c}{x}+\dfrac{a^2}{x^2}}$; whoſe Fluxion, which is $2\dot{x}\times\overline{b+x}\times\overline{1-\dfrac{2c}{x}+\dfrac{a^2}{x^2}}+\overline{b+x}|^2\times\overline{\dfrac{2c\dot{x}}{x^2}-\dfrac{2a^2\dot{x}}{x^3}}$, being put $=0$, and the whole Equation divided by $2\dot{x}\times\overline{b+x}$, there will come out $1-\dfrac{2c}{x}+\dfrac{a^2}{x^2}+\overline{b+x}\times\overline{\dfrac{c}{x^2}-\dfrac{a^2}{x^3}}=0$; whence $x^3-2cx^2+a^2x+\overline{b+x}\times\overline{cx-a^2}=0$; that is, (by Reduction) $x^3-cx^2+bcx-a^2b=0$: From the Reſolution of which Equation, the Poſition of DE is determined.

LEMMA.

35. *If a Body or Point* (n) *be ſuppoſed to move in a Right-line* AB, *its abſolute Celerity, in the Direction of that Line, will be to the relative Celerity, whereby it tends to, or from, a given Point* C, *any where out of the Line, as the Diſtance* Cn, *is to the Diſtance* Dn, *intercepted by* n *and the Perpendicular* CD; *or, as Radius to the Co-ſine of the Angle of Inclination* DnC.

For, putting CD $=a$, D$n=x$, and C$n=y$, we have $a^2+x^2=y^2$, and conſequently $2x\dot{x}=2y\dot{y}$ *: Whence

* Art. 2 and 5.



Whence $\dot{x}$: $\dot{y}$ :: y (Cn) : x (Dn) :: Radius : Co-ſine DnC : But, the Fluxions of Quantities are as the Celerities of their Increaſe *, therefore the Truth of the Propoſition is manifeſt.

Art. 2 and 5.

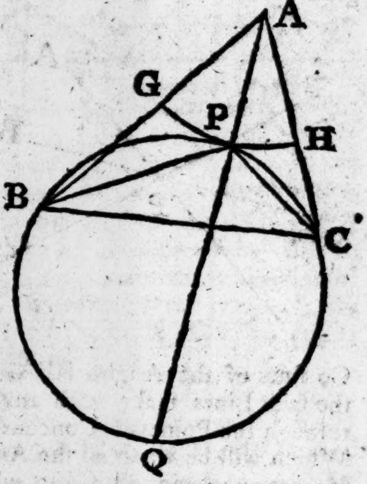
COROLLARY.

It follows from hence, that the relative Celerities in any two different Directions nE and nC, are directly as the Co-ſines of the correſponding Angles DnE and DnC. Therefore, when nE is perpendicular to Cn, (and the Angle DnE therefore equal to C) the Celerity in the Direction nE, will be to that in the Direction nC, as the Sine of DnC is to its Co-ſine. From whence it appears, that the Celerities in the Directions Dn, Cn, and En (perpendicular to nC) are to each other as Cn, Dn, and CD reſpectively.

EXAMPLE XII.

36. *To determine the Poſition of a Point, from whence, if three Right-lines be drawn to ſo many given Points* A, B, C, *their Sum ſhall be the leaſt poſſible.*

Let HPG be the Periphery of a Circle deſcribed about the Point A, as a Center, at any Diſtance AG; in which let the Point P be conceived to move with an uniform Celerity, from G towards H. Then, becauſe the relative Celerity thereof, in the Direction PC, is to that in the Direction BP produced, as the Co-ſine of the Angle CPH to the Co-ſine of the Angle BPG, *(by the preceding Lemma)*; and, ſince theſe Celerities, when the Sum of CP and BP is a *Minimum*, muſt be equal *, it follows, therefore, that the ſaid Angles CPH and BPG, as well as their Co-ſines, will in that Circumſtance become equal to each other; and conſequently APC alſo equal to APB. From whence it appears, that (take AG what you will) the Sum of the three Lines, AP, BP, and CP, cannot be the leaſt poſſible when the Angles APB and APC are unequal. And, by the ſame Argument, it alſo appears that their Sum cannot be the leaſt poſſible, when the Angles BPA and BPC are unequal: Therefore, their Sum muſt be the leaſt poſſible, when all the three Angles about the Point P are equal to one another; provided the Caſe will admit of ſuch an Equality, or that no one of the Angles of the Triangle ABC is equal to, or greater than $\frac{1}{3}$ of 4 Right Angles (for otherwiſe, the Point P will fall in the obtuſe Angle): Hence this

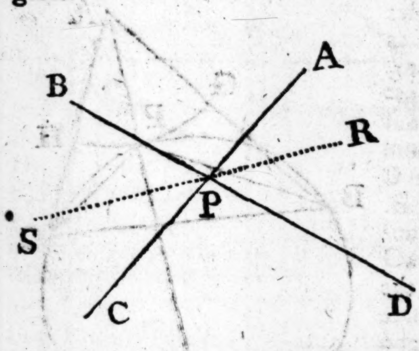


Construction.

Deſcribe, upon BC, a Segment of a Circle, to contain an Angle of 120°, and let the whole Circle BCQ be compleated; and from A, to the Middle (Q) of the Arch BQC, draw AQ interſecting the Circumference of the Circle in P; which will be the Point required. For, the Angles BPQ and CPQ, ſtanding upon the equal Arches BQ and CQ, have their Complements APB and APC equal to each other; and therefore, the Angle BPC being 120° (by Conſtruction) each of the ſaid

* Art. 2 and 22.

ſaid Angles APB, APC, will, likewiſe be 120 Degrees.

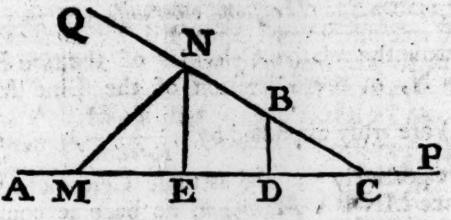


After the ſame Manner, it will appear that the Sum of all the Lines AP, BP, CP, &c. drawn from any Number of given Points A, B, C, &c. to meet in another Point P, will be the leaſt poſſible, when the Co-ſines of the Angles RPA, RPB, RPC, &c. that the ſaid Lines make with any other Line RS, paſſing through the Point of Concourſe, deſtroy each other: Which will be when all the Angles APB, BPC, CPD, &c. are equal, in all Caſes where the Poſition of the given Points will admit of ſuch an Equality. But, if the Number of given Points be four, the required Point will be in the Interſection of the two Right-lines joining the oppoſite Points: For, ſuppoſing APC and BPD to be continued Right-lines, the Co-ſine of RPA will be equal and contrary to that of RPC, and that of RPB equal and contrary to that of RPD.

EXAMPLE XIII.

37. *If two Bodies move at the ſame Time, from two given Places* A *and* B, *and proceed uniformly from thence in given Directions,* AP *and* BQ, *with Celerities in a given Ratio; it is propoſed to find their Poſition, and how far each has gone, when they are the neareſt poſſible to each other.*

Let M and N be any two cotemporary Poſitions of the Bodies, and upon AP let fall the Perpendiculars NE and BD; alſo let QB be produced to meet AP in



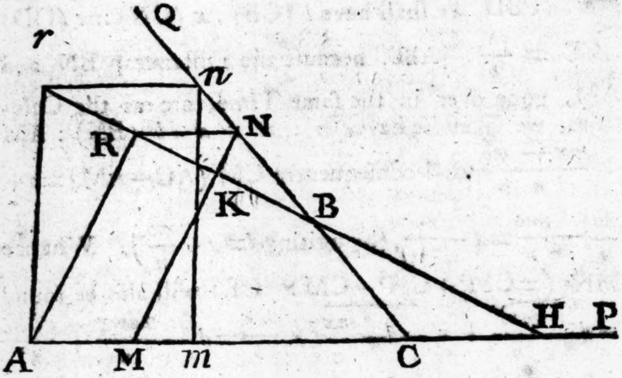
in C, and let MN be drawn: Moreover, let the given Celerity in BQ be to that in AP, as n to m, and let AC, BC, and CD, (which are alſo given) be denoted by a, b, and c reſpectively, and make the variable Diſtance CN $= x$: Then, by reaſon of the parallel Lines NE and BD, we ſhall have b (CB) : x (CN) :: c (CD) : CE $= \frac{cx}{b}$. Alſo, becauſe the Diſtances, BN and AM, gone over in the ſame Time, are as the Celerities, we likewiſe have, $n : m :: x - b$ (BN) : AM $= \frac{mx - mb}{n}$, and conſequently CM (AC—AM) $= a + \frac{mb}{n} - \frac{mx}{n} = d - \frac{mx}{n}$, (by writing $d = a + \frac{mb}{n}$). Whence MN² ($= \mathrm{CM}^2 + \mathrm{CN}^2 - \mathrm{CM} \times 2\mathrm{CE}$) will alſo be found $= \overline{d - \frac{mx}{n}}\big|^2 + x^2 - \overline{d - \frac{mx}{n}} \times \frac{2cx}{b} = d^2 - \frac{2dmx}{n} + \frac{m^2x^2}{n^2} + x^2 - \frac{2cdx}{b} + \frac{2cmx^2}{nb}$; whoſe Fluxion $- \frac{2dm\dot{x}}{n} + \frac{2m^2x\dot{x}}{n^2} + 2x\dot{x} - \frac{2cd\dot{x}}{b} + \frac{4cmx\dot{x}}{nb}$ being made $= 0$ (becauſe MN is to be a *Minimum*) we get $- bdmn + m^2bx + n^2bx - n^2cd + 2mncx = 0$; and conſequently $x = \frac{mnbd + n^2cd}{m^2b + n^2b + 2mnc} = \frac{nd \times \overline{mb + nc}}{b \times \overline{m^2 + n^2} + 2mnc}$; from whence BN, AM, and MN are alſo given.

The

The ſame otherwiſe.

Becauſe the relative Celerities of the two Bodies, at M and N, in the Direction of the Line MN (produced) are truly expreſſed by $\frac{\textit{Co-ſine}\ \text{M}}{\textit{Radius}} \times m$, and $\frac{\textit{Co-ſ.}\text{N}}{\textit{Rad.}}$
Art.35. $\times n$, reſpectively *; and as theſe Celerities, when the Diſtance MN is a *Minimum*, do become equal to each
Art.22. other †, it follows that, in this Circumſtance, $m : n ::$ Co-ſ. N. : Co-ſ. M : : Secant of M : Secant of N (*by plane Trig.*)

Whence this Conſtruction. Take CH to CB in the given Ratio of *m* to *n*, and draw HB; upon which

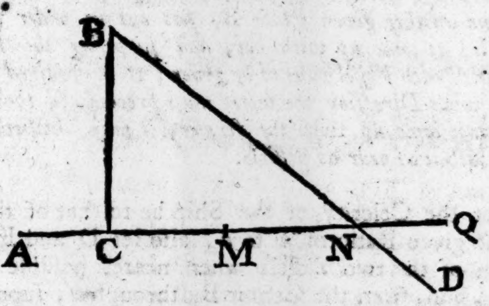


produced (if neceſſary) let fall the Perpendicular AR; draw RN parallel to AH, meeting CQ in N; laſtly, draw NM parallel to AR, and it will give the Poſition required. For, firſt, it is plain, becauſe AM (RN) : BN (: : CH : CB) : : *m* : *n*, that M and N are cotemporary Poſitions: It is likewiſe plain, that RN and BN will be Secants of the Angles KNR (CMN) and KNB (CNM) to the Radius NK; becauſe the Angle NKR (=ARK) is a Right-one. Which Lines or Secants are in the propoſed Ratio of *m* to *n*, as has been already ſhewn.

But the ſame Solution may be, yet, otherwiſe derived, independent of Fluxions, from Principles intirely geometrical. For, let *m* and *n* be any two cotemporary Poſitions at Pleaſure, and let CH (as before) be to CB, as the Celerity in AP to that in CQ; moreover, let *nr*, parallel to AP, be drawn, meeting HB produced in *r*, and let A, *r* be joined. Then, ſince CB : CH : : B*n* : *nr* (*by ſim. Triangles*) and CB : CH : : B*n* : A*m*, (*by Hyp.*) it follows, that *nr* and A*m*, (which are parallel) will alſo be equal to each other; and therefore A*r* and *mn*, likewiſe equal and parallel. But A*r* is the leaſt poſſible when perpendicular to H*r*. Whence the Solution is manifeſt.

EXAMPLE XIV.

38. *Let the Body* M *move, uniformly, from* A *towards* Q, *with the Celerity m, and let another Body* N *proceed from* B, *at the ſame time, with the Celerity n. Now it is propoſed to find the Direction* (BD) *of the latter, ſo that the Diſtance* MN *of the two Bodies, when the latter arrives in the Way or Direction* AQ *of the former, may be the greateſt poſſible.*



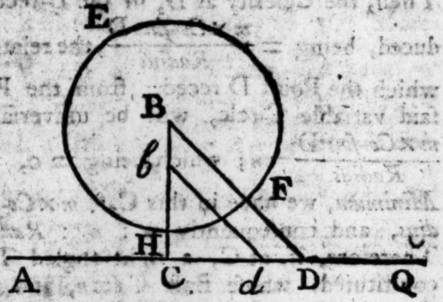
Let BC be perpendicular to AQ, and make AC = *a*, BC = *b*, and BN = *x*. Therefore, if the Poſition M be ſuppoſed cotemporary with N, we ſhall have $n : m :: x : \mathrm{AM} = \frac{mx}{n}$; whence $\mathrm{CM} = \frac{mx}{n} - a$, and conſequently MN (CN—CM) $= \sqrt{x^2 - b^2} - \frac{mx}{n} + a$; whereof the Fluxion being taken, and made = 0, we get $\frac{x}{\sqrt{x^2 - b^2}} = \frac{m}{n}$; therefore $x = \frac{mb}{\sqrt{m^2 - n^2}}$, and CN $(\sqrt{x^2 - b^2}) = \frac{nb}{\sqrt{m^2 - n^2}}$: Whence, $m : n$ (:: BN : CN :: Radius : Co-ſine N. The ſame Concluſion is otherwiſe derived, thus,

Let the Right-line BD be ſuppoſed to revolve about the Point B, as a Center, with a Motion ſo regulated, that the intercepted Part thereof BN may increaſe with the uniform Celerity *n*: Then, the Celerity with which
*Art. 35. CN is increaſed being $= \frac{n \times \textit{Radius}^{*}}{\textit{Co-ſine}\ \mathrm{N}}$, this Expreſſion, when MN is a *Maximum*, muſt, conſequently, be equal
†Art. 22. to (*m*) the Velocity of the other Body † M; and therefore $m : n$ :: Radius : Co-ſine N, as before.

EXAMPLE XV.

39. *Suppoſing a Ship to ſail from a given Place* A, *in a given Direction* AQ, *at the ſame time that a Boat, from another given Place* B, *ſets out in order (if poſſible) to come up with her, and ſuppoſing the Rate at which each Veſſel runs to be given; it is required to find in what Direction the latter muſt proceed, ſo that, if it cannot come up with the former, it may, however, approach it as near as poſſible.*

Let the Celerity of the Ship be to that of the Boat in the given Ratio of *m* to *n*; alſo let D and F be the Places of the two Veſſels when neareſt poſſible to each other, and, from the Center B, through F, ſuppoſe the Circumference of a Circle to be deſcribed. Then (the Diſtance DF being the leaſt poſſible), the Point F muſt be in the Right-line (DB) joining the Point D and the Center



Center B; becauſe no other Point in the whole Periphery, at which the Boat from B might arrive in the ſame time, is ſo near to D as that wherein the Line DB interſects the ſaid Periphery.—But now, to get an Expreſſion for DF, in algebraic Terms, let BC be perpendicular to AQ, and make AC $=a$, BC $=b$, and CD $=x$; and then BD $(\sqrt{BC^2+CD^2})$ will be $=\sqrt{b^2+x^2}$; moreover, becauſe $m:n::$ AD $(a+x)$: BF, you will have BF $=\frac{na+nx}{m}$, and conſequently, DF $=\sqrt{b^2+x^2}-\frac{na+nx}{m}$; whoſe Fluxion, $\frac{x\dot{x}}{\sqrt{b^2+x^2}}-\frac{n\dot{x}}{m}$, being made $=0$, we find $x=\frac{nb}{\sqrt{m^2-n^2}}$; whence the Direction BD is known: And, if the Value of x, thus found, be ſubſtituted in that of DF, (found above) we ſhall have DF $=\frac{b\sqrt{m^2-n^2}-na}{m}$; whence the Poſition of F is known. And from which it is obſervable, that, as DF muſt be a *real*, *poſitive* Quantity (by the Queſtion) this Method of Solution can only obtain when m is greater than n, and $b\sqrt{m^2-n^2}$, alſo greater than na: For in all other Caſes the Boat will be able to come up with the Ship.

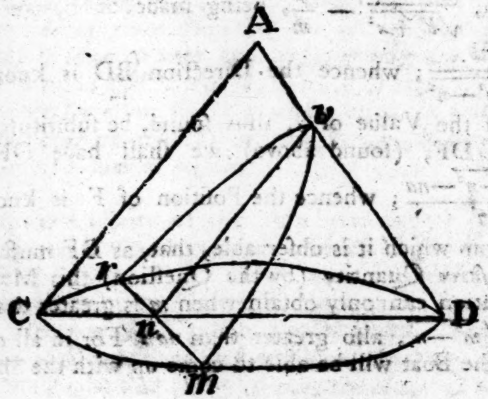
The ſame otherwiſe.

Let the Radius of the Circle EFH be conceived to increaſe uniformly, with the Celerity n, whilſt the Point

D moves uniformly along AQ, with the Celerity m: Then, the Celerity at D, in the Direction of BD produced, being $= \frac{m \times \textit{Co-ſine}\ D}{\textit{Radius}}$, the relative Celerity with which the Point D recedes from the Periphery of the ſaid variable Circle, will be univerſally expreſſed by $\frac{m \times \textit{Co-ſine}\ D}{\textit{Radius}} - n$; which being $= 0$, when DF is a *Minimum*, we have in this Caſe $m \times \textit{Co-ſine}\ D = n \times \textit{Radius}$, and conſequently $m : n :: \textit{Radius} : \textit{Co-ſine}\ D$. Therefore, if, at C, a right-angled Triangle Cbd be conſtituted, whoſe Baſe $Cd = n$, and its Hypothenuſe $db = m$, and parallel to the latter you draw BD, it will be the Direction required: In which, if there be taken BF, a Fourth-proportional to m, n, and AD, you will alſo have the Poſition required.

EXAMPLE XVI.

40. *To determine the greateſt Parabola that can be formed by cutting a given Cone* ACD.



Let mv, parallel to CA, be the Axis of the Parabola rvm, and rm the Baſe (or Ordinate) thereof; putting

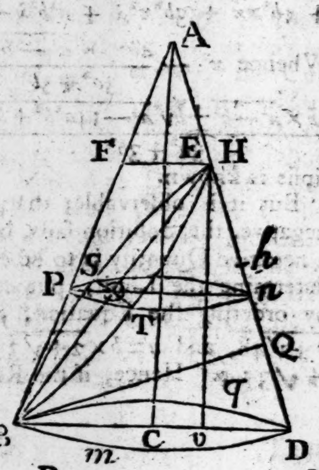
DC $=a$, CA $=b$, and D$n=x$; then, becauſe of the parallel Lines, it will be, $a:b::x:\frac{bx}{a}=nv$: Moreover, by the Property of the Circle, we have rn^2 $(=nm^2=\mathrm{D}n\times\mathrm{C}n)=ax-x^2$, and conſequently rm $2\sqrt{ax-x^2}$; which multiplied by $\frac{2}{3}\times\frac{bx}{a}$ (becauſe every Parabola is $\frac{2}{3}$ of a Parallelogram of the ſame Baſe and Altitude) gives $\frac{4bx}{3a}\sqrt{ax-x^2}$ for the Content of the Parabola: Whoſe Fluxion, or that of ax^3-x^4 * being put equal to Nothing; we find $x=\frac{3a}{4}$: Whence $nv=\frac{3}{4}\times\mathrm{AC}$, $rm=\mathrm{CD}\times\sqrt{\frac{3}{4}}$, and the Area of the greateſt, or required, Parabola $=\mathrm{AC}\times\mathrm{CD}\times\frac{\sqrt{3}}{4}$.

*Art. 26.

EXAMPLE XVII.

41. *To determine the greateſt Ellipſis* BTES *that can be formed by cutting a given Cone* ABD.

Let BE be the greater, and TS the leſſer, Axis of the Ellipſis BTES, conſidered as variable by the Motion of (the End of the Tranſverſe) E, along the Line AD; moreover let Ev be parallel to AC the Axis of the Cone, meeting the Diameter BD in v, and let the Diameters EF and np be parallel to BD; whereof the latter np is ſuppoſed to



paſs through O the Center of the Ellipſis: Then, putting $AC=a$, $CD=b$, and $Cv=x$, we ſhall have $Bv=b+x$; alſo, becauſe of the parallel Lines we have CD (b) : CA (a) :: Dv $(b-x)$: $\frac{a\times\overline{b-x}}{b}=Ev$; whence BE $(\sqrt{Bv^2+Ev^2})=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2}}{b}$. Furthermore, ſince the Triangles EOn, EBD, and BOp, BEF are equiangular, and EO $(=BO)=\frac{1}{2}$ BE, we likewiſe have $On=\frac{1}{2}BD=b$, and $Op=\frac{1}{2}EF=Cv=x$; and conſequently $On\times Op$ $(=OT^2$, *by the Property of the Circle*$)=bx$; whence $ST=2\sqrt{bx}$, and therefore $BE\times ST=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2\times 4bx}}{b}$.

Now the Area of any Ellipſis being in a conſtant Ratio to the Rectangle of its greater and leſſer Axes (namely as 3,14159, &c. to 4) the laſt general Expreſſion muſt therefore be a *Maximum*, when the Area is ſo; and therefore its Fluxion, or that of $b^2x\times\overline{b+x}|^2+a^2x\times\overline{b-x}|^2$ $(=b^4x+2b^3x^2+b^2x^3+a^2b^2x-2a^2bx^2+a^2x^3)$ equal to Nothing*; that is, $b^4\dot{x}+4b^3x\dot{x}+3b^2x^2\dot{x}+a^2b^2\dot{x}-4a^2bx\dot{x}+3a^2x^2\dot{x}=0$: Whence $x^2-\frac{4bx\times\overline{a^2-b^2}}{3a^2+3b^2}=-\frac{b^2}{3}$, and $x=\frac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; from which the Ellipſis is known.

But it is obſervable, that, when $a^4-14a^2b^2+b^4$ is negative, this Solution fails, becauſe the Square Root of a negative Quantity is to be extracted. Therefore, to determine the Limit, put $a^4-14a^2b^2+b^4=0$; then, by ordering the Equation, you will get $a^2=b^2\times\overline{7+\sqrt{48}}$, and $a=b\times\overline{2+\sqrt{3}}$; and therefore $a:b::2+\sqrt{3}:1$. Hence, if the Ratio of AC to CD be not greater than that of $2+\sqrt{3}$ to 1, or (which comes to the ſame thing) if the Angle DAC be not leſs than 15 Degrees, the Fluxion of the Ellipſis can never become equal to Nothing; but the Ellipſis itſelf will increaſe continually, from the Vertex till it coincides with the Baſe of the Cone; and therefore is greater at the Baſe than in any other Poſition.

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*Art.22.

But it is further to be obſerved, that this Problem is confined to, yet, narrower Limits. For, either the Ellipſis will increaſe, continually, from the Vertex, to the Baſe, of the Cone, (which is ſhewn to be the Caſe when the Angle DAC is greater than 15°) or elſe it will increaſe till the Point E arrives at a certain Poſition H, and afterwards decreaſe to another certain Poſition *b*, and then increaſe again till it coincides with the Baſe of the Cone, (for it muſt always increaſe again before it coincides with the Baſe, becauſe, after the Point E is got below the Perpendicular BQ, both the Axes of the Ellipſis increaſe at the ſame time),

The ſame thing alſo appears from the foregoing Equation $x = \frac{2b \times \overline{a^2 - b^2} \pm b\sqrt{a^4 - 14a^2b^2 + b^4}}{3a^2 + 3b^2}$; whoſe two Roots expreſs the two Values of x (or Cv) at the Times of the *Maximum* (at H) and its ſucceeding *Minimum* (at *b*). Hence it is manifeſt, that the Ellipſis may admit of a *Maximum* between the Vertex of the Cone and the Perpendicular BQ, and yet, that *Maximum* be leſs than the Baſe of the Cone, unleſs the foreſaid Angle DAC be ſo much leſs than 15° (above found) that the Increaſe from *b* to D, be leſs than the Decreaſe from H to *b*. Now therefore, to determine the exact Limit, let the foreſaid Increment and Decrement be ſuppoſed equal to each other, or, which is the ſame in Effect, let the Ellipſis BTESB = the Circle BqDm, or BE × ST = BD², that is, let $\frac{\sqrt{b^2 \times \overline{b+x}|^2 + a^2 \times \overline{b-x}|^2 \times \overline{4bx}}}{b} = 4b^2$: From which

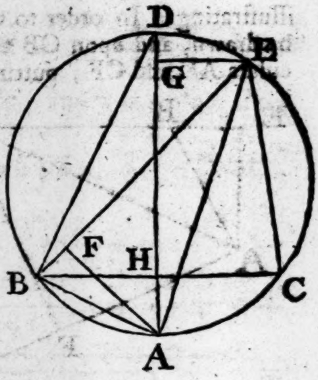
Equation you will get $a^2 = \frac{b^2}{x} \times \frac{4b^3 - b^2x - 2bx^2 - x^3}{\overline{b-x}|^2}$ $= \frac{b^2}{x} \times \frac{4b^2 + 3bx + x^2}{b - x}$: Moreover, from the Equation $\dot{b^4x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$, (given above) you will, again, get $a^2 = \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{-b^2 + 4bx - 3x^2}$ $= \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{\overline{b-x} \times \overline{3x-b}}$; Whence, by comparing theſe equal Values, there ariſes $\frac{4b^2 + 3bx + x^2}{x} = \frac{b^2 + 4bx + 3x^2}{3x - b}$ which, ordered, gives $x^2 + 2bx - b^2 = 0$, and therefore $x = b\sqrt{2} - b$.

Moreover, $\frac{a^2}{b^2}$ being $= \frac{4b^2 + 3bx + x^2}{bx - x^2}$, if $b^2 - 2bx$ be ſubſtituted herein for, its Equal, x^2, it will become $\frac{a^2}{b^2} = \frac{5b^2 + bx}{bx - x^2} = \frac{5b + x}{3x - b} = \frac{5b + b\sqrt{2} - b}{3b\sqrt{2} - 3b - b} = \frac{4 + \sqrt{2}}{-4 + 3\sqrt{2}}$ $= \frac{\overline{4 + \sqrt{2}} \times \overline{4 + 3\sqrt{2}}}{\overline{-4 + 3\sqrt{2}} \times \overline{4 + 3\sqrt{2}}} = \frac{22 + 16\sqrt{2}}{2} = 11 + 8\sqrt{2}$. Hence we have, $1 : \sqrt{11 + 8\sqrt{2}} :: b$ (DC) : a (AC) :: Radius to the Tangent of the Angle ADC = 78° 3′: Whoſe Complement DAC = 11° 57′, is the leaſt Limit poſſible. Therefore, unleſs the Angle which the ſlant Side makes with the Axis be leſs than 11° 57′, the greateſt Ellipſis will be leſs than the Baſe of the Cone.

EXAMPLE XVIII.

42. *Of all Triangles, having the ſame given Perimeter, and inſcribed in the ſame given Circle; to determine the greateſt.*

Let the Diameter DA biſect the Baſe BC of the required Triangle BEC in H, draw AE, AB and BD; alſo draw AF perpendicular to BE, and GE, parallel to BC,



BC, meeting AD in G: Then, putting AD = a, half the given Perimeter of the Triangle = b, and DH = y; we have BH = $\sqrt{ay-y^2}$, and therefore EF = $b-\sqrt{ay-y^2}$[a]. Moreover DH (y) : AD (a) :: DB² : DA² :: EF² $\left(\overline{b-\sqrt{ay-y^2}}\,^2\right)$: EA² $= \frac{a}{y} \times \overline{b-\sqrt{ay-y^2}}\,^2$; therefore AG $\left(\frac{AE^2}{AD}\right) = \frac{\overline{b-\sqrt{ay-y^2}}\,^2}{y}$, and HG = (AG—AH =) $\frac{b^2-2b\sqrt{ay-y^2}}{y}$; whence the Area of the Triangle BEC (BH × HG) $= \frac{b^2\sqrt{ay-y^2}}{y} - 2ba + 2by$, whoſe Fluxion $2b\dot{y} - \frac{\frac{1}{2}ab^2\dot{y}}{y\sqrt{ay-yy}}$ being put = 0, gives $y\sqrt{ay-yy} = \frac{1}{4}ba$; whence y, and from thence the Sides of the Triangle may be determined.

EXAMPLE XIX.

43. *To determine the greateſt Area that can be contained under four given Right-lines.*

Though it is demonſtrable from common Geometry that the Area will be a *Maximum*, when the Trapezium ABCD, formed by the given Lines, may be inſcribed in a Circle[b], yet I ſhall here give the Solution from the Principles of Fluxions, (whoſe Uſes I am now illuſtrating).

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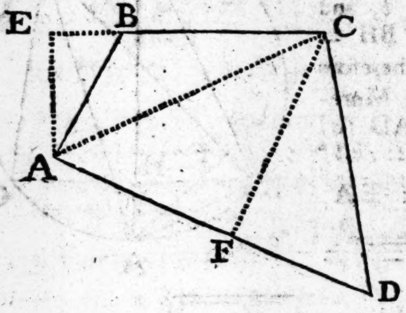
[a] By Prop. 13. Page 62. *Elem. Trig.*

[b] See Page 117. of *Elem. Geometry*.



illuſtrating. In order to which, let the Diagonal AC be drawn, and upon CB and AD let fall the Perpendiculars AE and CF; putting AB $=a$, BC $=b$, CD $=c$, DA $=d$, BE $=x$, and DF $=y$: Then AE being $=\sqrt{a^2-x^2}$, and CF $=\sqrt{c^2-y^2}$, the Area of the Trapezium ($\frac{1}{2}$BC × AE + $\frac{1}{2}$AD × CF) will be $=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{1}{2}d\sqrt{c^2-y^2}$; *Art.22. and its Fluxion $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{a^2-x^2}}-\frac{\frac{1}{2}dy\dot{y}}{\sqrt{c^2-y^2}}=0$ *; and therefore $\frac{-dy\dot{y}}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$. Moreover, ſince b^2+a^2+2bx ($=$AC^2) $=d^2+c^2-2dy$, by taking the Fluxion thereof, we have $2b\dot{x}=-2d\dot{y}$, or $-d\dot{y}=b\dot{x}$; which, ſubſtituted for $-d\dot{y}$ in the foregoing Equation, gives $\frac{b\dot{x}y}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$, and $\frac{y}{\sqrt{c^2-y^2}}=\frac{x}{\sqrt{a^2-x^2}}$; and conſequently, $\sqrt{c^2-y^2}$ (CF) : y (DF) :: $\sqrt{a^2-x^2}$ (AE) : x (BE) : From which it appears that the Triangles DCF and ABE are ſimilar, and that (D + ABC being = 2 Right-angles) the Trapezium may be inſcribed in a Circle; but this by the Bye. We are now to get an Expreſſion for the Area in known Terms, and in order thereto we have $b^2+a^2+2bx=dd+c^2-2dy$, $y=\frac{cx}{a}$, and CF $=\frac{c\sqrt{a^2-x^2}}{a}$ (becauſe AB : BE :: DC : DF, &c.) : Therefore, by Subſtitution, $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, and the Area ($\frac{1}{2}$BC × AE

+ $\frac{1}{2}$AD

$+\frac{1}{2}\mathrm{AD}\times\mathrm{CF}) = \frac{1}{2}b\sqrt{a^2-x^2}+\frac{cd}{2a}\sqrt{a^2-x^2} = \frac{ab+cd}{2a}\sqrt{a^2-x^2}$; and therefore the Square thereof $= \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a^2-x^2} = \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a+x}\times\overline{a-x} = \frac{\overline{ab+cd}|^2}{4}\times\overline{1+\frac{x}{a}}\times\overline{1-\frac{x}{a}}$. But ſince $b^2+a^2+2bx = d^2+c^2-\frac{2cdx}{a}$, we have $\frac{x}{a} = \frac{d^2+c^2-b^2-a^2}{2ab+2cd}$, $1+\frac{x}{a} = 1+\frac{dd+c^2-b^2-a^2}{2ab+2cd} = \frac{2ab+2cd+dd+c^2-b^2-a^2}{2ab+2cd} = \frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}$; and $1-\frac{x}{a} = \frac{2ab+2cd-dd-c^2+b^2+a^2}{2ab+2cd} = \frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd}$; and conſequently, the Square of the Area $= \frac{\overline{ab+cd}|^2}{4}\times\frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}\times\frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd} = \frac{\overline{\overline{d+c}|^2-\overline{b-a}|^2}\times\overline{\overline{b+a}|^2-\overline{a-c}|^2}}{16}$ which (becauſe the Difference of the Squares of any two Quantities is equal to a Rectangle under their Sum and Difference) will alſo be $= \frac{\overline{d+c+b-a}\times\overline{d+c-b+a}\times\overline{b+a+a-c}}{4}\times\frac{\overline{b+a-d+c}}{4} = \overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-a}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-b}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-d}$. Whence it appears, that, if from $\frac{1}{2}$ the Sum of all the four Sides each particular Side be ſubtracted, the continual Product of the Remainders will be the Square, or ſecond Power, of the Area.

From this Theorem, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is deduced, as a Corollary: For, making $a=0$,

$a=0$, the Trapezium becomes a Triangle, and the ſecond Power of its Area $=\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-b}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-c}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-d}$: Which, in Words, is the common Rule.

EXAMPLE XX.

44. *To find the greateſt Value of y in the Equation* $a^4x^2=\overline{xx+yy}|^3$.

By putting the whole Equation into Fluxions, &c. we have $2a^4x\dot{x}=\overline{2x\dot{x}+2y\dot{y}}\times3\times\overline{x^2+y^2}|^2$; which in the required Circumſtance, when $\dot{y}=0$ *, becomes $2a^4x\dot{x}=6x\dot{x}\times\overline{x^2+y^2}|^2$; whence $x^2+y^2=\frac{a^2}{\sqrt{3}}$, and $\overline{x+y^2}|^3=\frac{a^6}{3\sqrt{3}}$: But, by the given Equation $\overline{x^2+y^2}|^3=a^4x^2$; conſequently $a^4x^2=\frac{a^6}{3\sqrt{3}}$, and therefore $x=a\sqrt{\frac{1}{3\sqrt{3}}}$; whence $y^2\left(=\frac{a^2}{\sqrt{3}}-x^2\right)=\frac{2a^2}{3\sqrt{3}}$, and $y=a\sqrt{\frac{2}{3\sqrt{3}}}$.

* Art. 22.

The ſame otherwiſe.

Since $\overline{xx+yy}|^3$ is given $=a^4x^2$, we have $x^2+y^2=a^{\frac{4}{3}}\times x^{\frac{2}{3}}$, and therefore $y^2=a^{\frac{4}{3}}\times x^{\frac{2}{3}}-x^2$; whoſe Fluxion, $\frac{2}{3}a^{\frac{4}{3}}\times x^{-\frac{1}{3}}\dot{x}-2x\dot{x}$, being put $=0$, we alſo get $\frac{a^{\frac{4}{3}}\times x^{-\frac{1}{3}}}{3}=x$; whoſe Cube is $\frac{a^4\times x^{-1}}{27}=x^3$, or $\frac{a^4}{27x}=x^3$; whence $27x^4=a^4$, and conſequently $x\pm a\sqrt{\frac{1}{3\sqrt{3}}}$, the ſame as before.

45. When, in the general Expreſſion, whoſe *Maximum* or *Minimum* is ſought, there are two or more indeterminate Quantities, independent of each other, their reſpective Values, in the required Circumſtance, will be determined, by making them flow, one by one, while the others are ſuppoſed invariable; as in the following

EXAMPLE XXI.

Wherein it is propoſed to find three ſuch Values of x, y, and z, as ſhall make the Value of $\overline{b^3-x^3}\times\overline{x^2z-z^3}\times\overline{xy-y^2}$ *the greateſt poſſible.*

Firſt, conſidering y as variable, and the reſt conſtant, we have $x\dot{y}-2y\dot{y}=0$ *; whence $y=\frac{1}{2}x$, and $xy-y^2=\frac{1}{4}x^2$. By making z variable, we have $x^2\dot{z}-3z^2\dot{z}=0$; whence $z=\frac{x}{\sqrt{3}}$, and $x^2z-z^3=\frac{2x^3}{3\sqrt{3}}$. Now let theſe Values of $xy-y^2$ and x^2z-z^3 be ſubſtituted in the given Expreſſion, and it will become $\frac{x^2}{4}\times\frac{2x^3}{3\sqrt{3}}\times\overline{b^3-x^3}=\frac{b^3x^5-x^8}{6\sqrt{3}}$; therefore $5b^3x^4\dot{x}-8x^7\dot{x}=0$: Whence $x=\frac{1}{2}b\times\sqrt[3]{5}$, $y\ (=\frac{1}{2}x)=\frac{1}{4}b\times\sqrt[3]{5}$, and $z\ \left(=\frac{x}{\sqrt{3}}\right)=\frac{1}{2}b\times\frac{\sqrt[3]{5}}{\sqrt{3}}$. *Art.22.

The Reaſon of the foregoing Proceſs is obvious: For, if the Fluxion of the given Expreſſion, when any one of the indeterminate Quantities is made variable, be not equal to Nothing, that Expreſſion may become greater, without altering the Values of the reſt, which are conſidered as conſtant †: And therefore cannot be the greateſt poſſible, unleſs the ſaid Fluxion is equal to Nothing. †Art.22.

EXAMPLE XXII.

46. *To determine the different Values of x, when that of $3x^4 - 28ax^3 + 84a^2x^2 - 96a^3x + 48b^4$ becomes a* Maximum *or* Minimum.

The Fluxion of the given Expreſſion being *(as uſual)* put equal to Nothing, we have $12x^3 - 84ax^2 + 168a^2x - 96a^3 = 0$, or $x^3 - 7ax^2 + 14a^2x - 8a^3 = 0$: From whence (by the Method of Diviſors) we get $x - a = 0$, $x - 2a = 0$, or $x - 4a = 0$: Therefore, the Roots of the Equation, or the three Values of x, are a, $2a$, and $4a$.

SCHOLIUM.

47. It appears, from the laſt Example, that a Quantity may admit of as many *Maxima* and *Minima* (according to the Meaning of the Definition *) as there are poſſible Roots in the Equation, ariſing from aſſuming its Fluxion equal to Nothing. Now to know which of thoſe Roots point out a *Maximum*, and which a *Minimum*; find whether the Value of the ſaid Fluxion, a little before it becomes equal to Nothing, be poſitive or negative; if *poſitive*, the ſucceeding Root gives a *Maximum*; but if *negative*, a *Minimum*: The Reaſon of which is extremely obvious; becauſe ſo long as any Quantity increaſes, its Fluxion is poſitive, but when it decreaſes the Fluxion is negative.

*Art. 22.

As an Example hereof, let the Quantity $3x^4 - 28ax^3 + 84a^2x^2 - 96a^3x + 48b^4$, be again reſumed; whoſe Fluxion is $12\dot{x} \times \overline{x^3 - 7ax^2 + 14a^2x - 8a^3} = 12\dot{x} \times \overline{x - a} \times \overline{x - 2a} \times \overline{x - 3a}$: Whereof the Value, before it becomes equal to Nothing, the firſt time (or before $x = a$) being negative (becauſe the Product of three negative Factors is negative) its firſt Root (a) therefore indicates a *Minimum*: Whence we may conclude, without conſidering farther, that the ſecond Root $(2a)$ gives a *Maximum*, and the third $(4a)$ another *Minimum*. But, if you would know whether the firſt or third Root gives the leſſer Value of the two; it is but ſubſtituting in the given Quantity, which will come out $48b^4-37a^4$, and $48b^4-64a^4$ reſpectively; therefore the latter is the leſſer, and the very leaſt Value the propoſed Expreſſion can admit of.

When all the Roots prove impoſſible, the Quantity propoſed (as its Fluxion can never become $=0$) muſt either increaſe, or decreaſe, continually; and therefore can neither admit of a *Maximum nor a Minimum.*

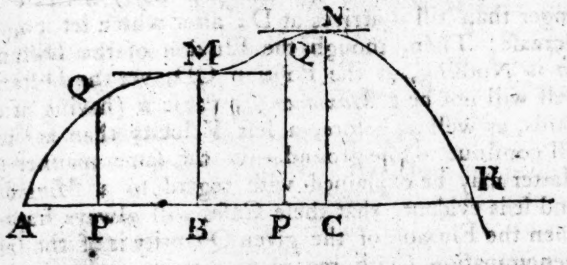
Moreover, it may ſo happen, that the Roots are poſſible, the Fluxion $=0$, and yet the Quantity itſelf be neither a *Maximum* nor a *Minimum* in that Circumſtance.

For let us, again, ſuppoſe the Point n to move after m, as in the general Illuſtration, (*vid. Art.* 22.) only let the Velocity of n (*in the firſt Caſe*) increaſe no longer than 'till it arrives at D; after which let it again decreaſe: Then, though the Fluxion of the Diſtance mn is Nothing, at the Poſition CD, yet the Diſtance itſelf will not be a *Maximum*; becauſe n (having afterwards, as well as before, a leſs Velocity than m) will ſtill continue to loſe ground.—In the ſame manner the Matter may be explained with regard to a *Minimum.* And it is evident, that theſe Caſes will always happen when the Fluxion of the given Quantity is of the ſame Denomination (with regard to poſitive and negative), both before and after, it becomes equal to Nothing: Which, by the Rules of common Algebra, is known to be when the Equation admits of an even Number of equal Roots.—An Example hereof, however, may not be improper.

Let then the Quantity propoſed be $24a^3x-30a^2x^2+16ax^3-3x^4$; whoſe Fluxion is $24a^3\dot{x}-60a^2x\dot{x}+48ax^2\dot{x}-12x^3\dot{x}=12\dot{x}\times\overline{a-x}\times\overline{a-x}\times\overline{2a-x}$: Which being made $=0$, it appears that the two leaſt Roots are equal. Therefore there is neither a *Maximum* nor *Minimum* when $x=a$ (becauſe whether x be taken a little leſs, or a little greater, than a, the Value of the Fluxion will ſtill be affirmative.) The greateſt Root, however, not being affected with another equal one, indicates a *Maximum*, according to the Rule above preſcribed.

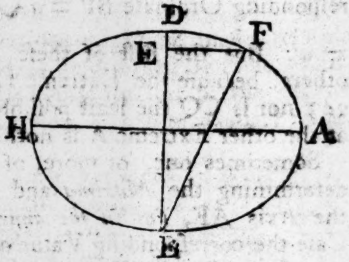
To render what has been obſerved above ſtill more conſpicuous, let the given Expreſſion, $24a^3x - 30a^2x^2 + 16ax^3 - 3x^4$, be repreſented by the variable Ordinate PQ of the Curve AQMNR; whoſe Abſciſſa AP is (as uſual) denoted by x.

Then, whilſt $(12\dot{x} \times \overline{a-x} \times \overline{a-x} \times \overline{2a-x})$ the Fluxion of the Ordinate continues poſitive, (or till x becomes $= a =$ AB) the Ordinate itſelf will increaſe: But at the Poſition BM it becomes ſtationary (if I may be allowed the Expreſſion) the Fluxion being then $= 0$. After which, the Fluxion being again affirmative, the Ordinate will again increaſe, till x becomes $= 2a$ ($=$ AC); when, the Fluxion becoming Nothing, a ſecond time,) and afterwards negative, CN will be a *Maximum*: Soon after which the Curve deſcends below its Axis, and continues to recede from it *in infinitum*.



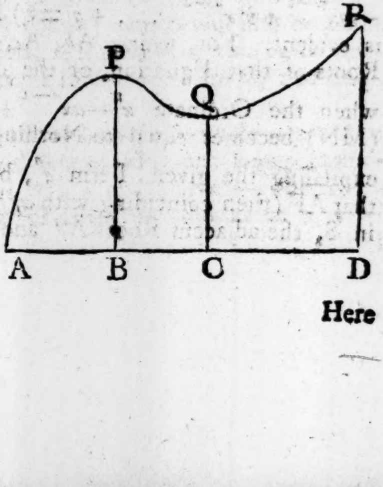
Another Thing there is that ought to be regarded in the Solution of theſe Kinds of Problems, and that is, whether the *Maxima* or *Minima*, found by aſſuming the Fluxion $= 0$, fall within the Limits preſcribed by the Nature of the Queſtion or Figure; which is often reſtrained by Conditions that do not enter into the algebraic Computation.

Thus, for Example; ſuppoſe it were required to find that Point (F) in a given Ellipſis ABHD which, of all others, is the moſt remote from the Extreme B of the conjugate Axis BD.



Then, drawing FE parallel to the Tranſverſe AH, and putting AH $= a$, BD $= b$, and BE $= x$, we have, by the Property of the Curve BF^2 $(= BE^2 + EF^2)$ $x^2 + \overline{bx - x^2} \times \frac{a^2}{b^2}$; from whence x is found $= \frac{\frac{1}{2}a^2b}{a^2 - b^2}$. But, from the Nature of the Figure, the greateſt Value that x $(= BE)$ can poſſibly admit of is b $(= BD)$, therefore if the Relation of a and b be ſuch, that $\frac{\frac{1}{2}a^2b}{a^2 - b^2}$ is greater than b, this Solution is manifeſtly impoſſible. —— To determine the Limit, therefore, make $\frac{\frac{1}{2}a^2b}{a^2 - b^2} = b$; then it will be found that $2b^2 = a^2$. Whence the foregoing Solution can only obtain when $2BD^2$ is equal to, or leſs than AH^2.

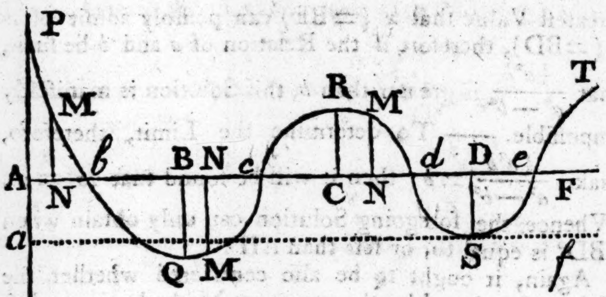
Again, it ought to be alſo conſidered whether the Value of x, found by the common Method, gives a leſs Quantity for the *Maximum*, and a greater for the *Minimum*, than will ariſe from the Extremes themſelves by which x is limited.



Thus, let it be required to determine the greateſt and leaſt Ordinates in a Curve, APR, whoſe Equation is $y^3 = 6a^2x - 9ax^2 + 4x^3$, and whoſe greateſt Abſciſſa AD is given equal $2a$.

Here we ſhall, by taking the Fluxion, &c. have $x = \frac{1}{2}a$, or $x = a$: The former of which Values gives the correſponding Ordinate $BP = a\sqrt[3]{\frac{5}{4}}$; and the latter, $CQ = a$: But the firſt of theſe is not the greateſt of all others, becauſe the Extreme DR exceeds it, being $= 2a$; nor is CQ the leaſt poſſible, becauſe the Ordinate at the other Extreme A is nothing at all.

Sometimes one, or more, of the Points Q, S, &c. determining the *Maxima* and *Minima*, will fall below the Axis AF, (*as in the annexed Figure*). In which Caſe the correſponding Value of the General Expreſſion, repreſented by the Ordinate, will be negative: But at the Points *b*, *c*, *d*, &c. where the Curve interſects the Axis, it will be equal to nothing: Whence (by the Bye) the Reaſon why the Roots of an Equation ($x^n - ax^{n-1} + b^2x^{n-2} \ldots + q^n = 0$) are impoſſible by Pairs is evident. For, ſeeing A*b*, A*c*, A*d*, A*e*, &c. are the Roots of that Equation, or the different Values of x, when the Ordinate $x^n - ax^{n-1} + b^2x^{n-2} \ldots\ldots + q^n$ (MN) becomes equal to Nothing, it is plain, if PA, expreſſing the given Term q^n, be increaſed to P*a*, ſo that AF (then coinciding with *af*) may touch the Curve in S, the adjacent Roots A*d* and A*e* will then become equal;



equal; and if q^n be fårther increaſed, ſo that the Axis may fall wholly below the Curve, not only thoſe two, but alſo the other Roots, A*b* and A*c*, will become impoſſible.

Various other Obſervations might be made, relating to the Limits of Equations, determined by theſe *Maxima* and *Minima*; but this being foreign to the Matter in hand, I ſhall content myſelf with one Remark more, *viz.*

Any Expreſſion which, being put equal to Nothing, admits of two or more equal Roots, has as many ſucceeding Orders of Fluxions equal to Nothing, at the ſame time, as are expreſſed by the Number of thoſe Roots minus *one.*

Thus, an Equation, having three equal Roots, has both its firſt and ſecond Fluxions equal to Nothing, when the Fluent itſelf is equal to Nothing.

Hence we have another Way (beſides that given above) to know when a Quantity may have its Fluxion equal to Nothing, and yet neither admit of a *Maximum* nor a *Minimum*: For, ſince this Circumſtance always takes place when the Equation admits of an *even* Number of equal Roots (as has been already ſhewn) the Number of Orders of Fluxions, equal to Nothing, at the ſame time (including the Firſt) muſt alſo be even.

Hence, alſo, we have an eaſy Method for diſcovering when ſome of the Roots of an Equation are equal; and, if ſo, what they are.

Thus, let $x^3 - 3ax^2 + 4a^3 = 0$ be propounded; whereof the Fluxion $3x^2\dot{x} - 6ax\dot{x}$ being aſſumed equal to Nothing, we find $x = 2a$; which will alſo be a Root of the given Equation, if it admits of two equal ones: To try it, therefore, I ſubſtitute $2a$ for x, and find it anſwers.

Again, let $8x^4 - 28ax^3 + 18a^2x^2 + 27a^3x - 27a^4 = 0$; whereof the firſt and ſecond Fluxions being $32x^3\dot{x} - 84ax^2\dot{x} + 36a^2x\dot{x} + 27a^3\dot{x}$ and $96x^2\dot{x}^2 - 168ax\dot{x}^2 + 36a^2\dot{x}^2$, if the latter of them be aſſumed $= 0$, x will be found $=\frac{7a}{8}\pm\sqrt{\frac{25a^2}{64}}=\frac{3a}{2}$, or $\frac{a}{4}$: One of which Quantities, if the Equation propoſed admits of three equal Roots, will be the Value of each of them: By trying $\frac{3a}{2}$, it will be found to ſucceed. Whence, by a well known Rule, the fourth Root (being $=\frac{28a}{8}-\frac{3a}{2}\times 3=-a$) is alſo given.

The Reaſon of theſe Operations, as well as what is aſſerted above, may be thus demonſtrated.

Let $\overline{r-x}\times\overline{r-x}$ *&c.* $\times\overline{A+Bx+Cx^2}$ *&c.* $=0$, be any Equation, having two or more equal Roots, repreſented, each, by r: Put $y=r-x$, and let the Number of the equal Roots be denoted by n; then, by Subſtitution, we have $y^n\times\overline{A+B\times\overline{r-y}+C\times\overline{r-y}\rvert^2}$ *&c.* $=0$; which, by expanding the Powers of $r-y$, and putting $a=A+Br+Cr^2$ *&c.* $b=B+2Cr+3Dr^2$, *&c.* will be further transformed to $y^n\times\overline{a-by+cy^2-dy^3}$ *&c.* $=$: Whoſe Fluxion $na\dot{y}y^{n-1}-\overline{n+1}\cdot b\dot{y}y^n+\overline{n+2}\cdot c\dot{y}y^{n+1}$ *&c.* is evidently equal to Nothing, when y, or its Equal $r-x$, is Nothing (provided n be greater than Unity. It is equally plain, that the ſecond Fluxion $n\cdot\overline{n-1}\cdot a\dot{y}^2y^{n-2}-\overline{n+1}\cdot nb\dot{y}^2y^{n-1}+\overline{n+2}\cdot\overline{n+1}\cdot c\dot{y}^2y^n$ *&c.* will alſo be equal to Nothing, in the ſame Circumſtance, if n be greater than 2. *&c. &c.*

Hence, univerſally, let the Number (n) of equal Roots be what it will, that of the Orders of Fluxions equal to Nothing, at the ſame time, will be expreſſed by that Number *minus* one, as was to be ſhewn.

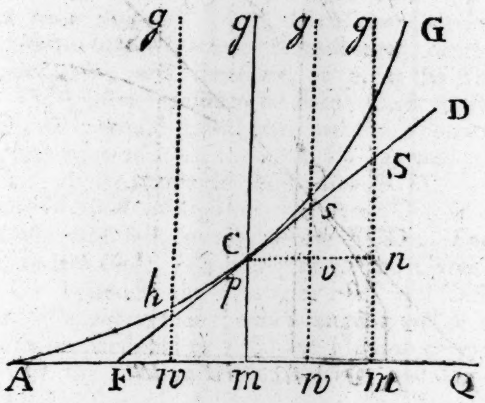
SECT.

SECTION III.

The Uſe of FLUXIONS *in drawing Tangents to Curves.*

ILLUSTRATION.

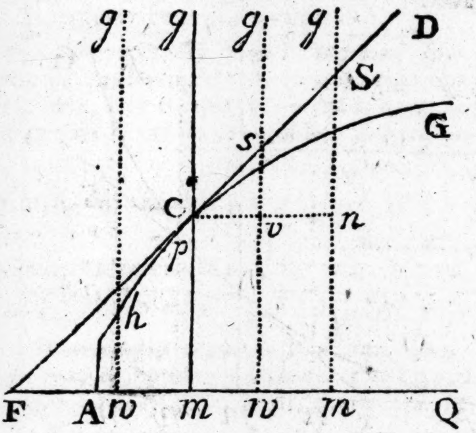
48. LET ACG be a Curve of any kind, and C the given Point from whence the Tangent is to be drawn.



Conceive a Right-line *mg* to be carried along uniformly, parallel to itſelf, from A towards Q, and let, at the ſame time, a Point *p* ſo move in that Line, as to deſcribe, or trace out, the given Curve ACG: Alſo let *mm*, or C*n* (equal and parallel to *mm*) expreſs the Fluxion of A*m*, or the Celerity wherewith the Line *mg* is carried; and let *n*S expreſs the correſponding Fluxion of *mp*, in the Poſition *m*C*g*, or the Celerity of the Point *p*, in the Line *mg*. Moreover, through the Point C let the Right-line SF be drawn, meeting the Axis of the Curve (AQ) in F.

Now, it is evident, if the Motion of *p*, along the Line *mg*, was to become equable at C, the Point *p* would be at S, when the Line itſelf had acquired the Poſition *m*S*g* (becauſe, by Hypotheſis, C*n* and *n*S expreſs the Diſtances that might be deſcribed by the two uniform Motions in the ſame time).

And, if *wsg* be aſſumed to repreſent any other Poſition of that Line, and *s* the contemporary Poſition of the Point *p* (ſtill ſuppoſing an equable Celerity of *p*); then the Diſtances C*v* and *vs*, gone over, in the ſame time, by the two Motions, will, always, be to each other as the Celerities, or as C*n* to *n*S: Therefore, ſince C*v* : *vs* :: C*n* : *n*S (which is a known Property of ſimilar Triangles) the Point *s* will, always, fall in the Right-line FCS: Whence it appears, that, if the Motion of the Point *p* along the Line *mg* was to become uniform at C, that Point would then move in the Right-line CS, inſtead of the Curve-line CG.



Now, ſeeing the Motion of *p*, in the Deſcription of Curves, muſt, either, be an accelerated or a retarded one, let it be, firſt, conſidered as an accelerated one: In which Caſe the Arch CG will fall, *wholly*, above the Right-line CD (as in Fig. 1.) becauſe the Diſtance of the Point *p* from the Axis AQ, at the End of any given Time, is greater than it would be if the Acceleration was to ceaſe at C; and, if the Acceleration had ceaſed at C, the Point *p* would (it is proved) have been always found in the ſaid Right-line FS.

But if the Motion of the Point *p* be a retarded one, it will appear, by reaſoning in the ſame manner, that the Arch CG will fall wholly below the Right-line CD (as in Fig. 2.)

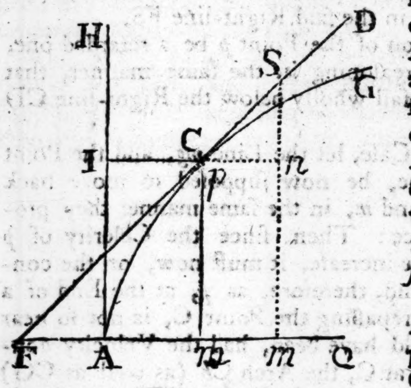
This being the Caſe, let the Line *mg*, and the Point *p*, along that Line, be now ſuppoſed to move back again, towards A and *m*, in the ſame manner they proceeded from thence: Then, ſince the Celerity of *p* (*Fig.* 1.) did before increaſe, it muſt now, on the contrary, decreaſe; and, therefore, as *p*, at the End of a given Time, after repaſſing the Point C, is not ſo near to AQ, as it would have been, had the Velocity continued the ſame as at C, the Arch C*b* (as well as CG) muſt fall wholly above the Right-line FCD. And, by the ſame Method of arguing, the Arch C*b*, *in the ſecond Caſe*, will fall, *wholly*, below FCD: Therefore FCD, *in both Caſes*, is a Tangent to the Curve at the Point C: Whence, the Triangles F*m*C and C*n*S being ſimilar, it appears, that the Sub-tangent *m*F is always a Fourth-proportional to (*n*S) the Fluxion of the ordinate (C*n*), the Fluxion of the Abſciſſa, and the Ordinate (C*m*).

Otherwiſe.

49. Let ACG repreſent the propoſed Curve, and let the Right-line FCD be a Tangent to it, at any Point C, meeting the Axis AQ (produced if neceſſary) in F: Suppoſe a Point *p* to move along the Curve, from A towards G, and let the abſolute Celerity thereof at C, in the Direction of the Tangent CD, or the Fluxion of the Line A*p* ſo generated *, be denoted by CS, any Part of the ſaid Tangent: Then, if AH, *mp* and *m*S be made perpendicular, and I*pn* parallel, to AQ, the relative Celerities of that Point, in the Directions C*n* and *m*C, wherewith I*p* (= A*m*) and *mp* increaſe in this

* Art. 2 and 5.

*Art. 35. Poſition, will be truly expreſſed by Cn and nS *: But the Celerities by which Quantities increaſe are as the Fluxions of thoſe Quantities: Therefore (CS being the Fluxion of the Curve-line Ap) Cn and nS are the correſponding Fluxions of the Abſciſſa Am and the Ordinate mp: And we have $Sn : nC :: mC : mF$, *the ſame as before.*

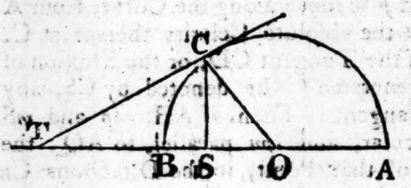


Hence, if the Abſciſſa Am be put $= x$, and the Ordinate $mp = y$, we ſhall have $mF = \frac{y\dot{x}}{\dot{y}}$: By means of which general Expreſſion, and the Equation expreſſing the Relation between x and y, the Ratio of the Fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the Length of the Sub-tangent (mF) as in the following Examples.

EXAMPLE I.

50. *To draw a Right-line* CT, *to touch a given Circle* BCA, *in a given Point* C.

Let CS be perpendicular to the Diameter AB, and put $AB = a$, $BS = x$ and $SC = y$: Then, by the Property of the Circle, y^2 $(CS^2) = BS \times AS$ $(= x \times \overline{a-x})$ $= ax - x^2$; whereof



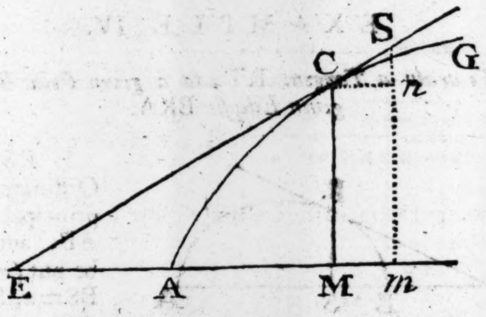
whereof the Fluxion being taken, in order to determine the Ratio of $\dot{x}$ and $\dot{y}$, we get $2y\dot{y} = a\dot{x} - 2x\dot{x}$; conſequently $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a - 2x} = \frac{y}{\frac{1}{2}a - x}$, which, multiplied by y, gives $\frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\frac{1}{2}a - x} =$ the Sub-tangent ST *. Whence (O being ſuppoſed the Center) we have OS $(\frac{1}{2}a - x)$: CS (y) :: CS (y) : ST; which we alſo know from other Principles.

*Art. 48 and 49.

EXAMPLE II.

51. *To draw a Tangent to any given Point* C *of the conical Parabola* ACG.

If the *Latus Rectum* of the Curve be denoted by a, the Ordinate MC by y, and its correſponding Abſciſſa



AM by x; then the known Equation, expreſſing the Relation of x and y, being $ax = y^2$, we have, in this Caſe, $a\dot{x} = 2y\dot{y}$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a}$, and conſequently $\frac{y\dot{x}}{\dot{y}}$† $= \frac{2y^2}{a} = \frac{2ax}{a} = 2x =$ MF. Therefore the Sub-tangent is juſt the double of its correſponding Abſciſſa AM: Which we likewiſe know from other Principles.

† Art. 48 and 49.

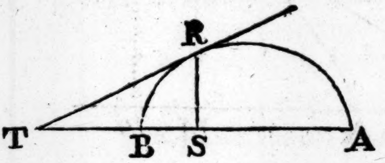
EXAMPLE III.

52. *To draw a Tangent to a Parabola of any kind.*

The general Equation of theſe ſort of Curves being $a^m x^n = y^{m+n}$, we have $na^m x^{n-1}\dot{x} = \overline{m+n} \times y^{m+n-1}\dot{y}$, and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n-1}}{na^m x^{n-1}}$; whence $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n}}{na^m x^{n-1}} = \frac{\overline{m+n} \times a^m x^n}{na^m x^{n-1}}$ (becauſe $y^{m+n} = a^m x^n$) $= \frac{m+n}{n} \times x =$ the true Value of the Subtangent: Which, therefore, is to the Abſciſſa, in the conſtant Ratio of $m+n$ to n.

EXAMPLE IV.

53. *To draw a Tangent* RT, *to a given Point* R, *in a given Ellipſis* BRA.



If RS be an Ordinate to the principal Axis AB, and there be put *(as uſual)* BS $= x$, RS $= y$, AB $= a$, and the leſſer Axis $= b$; we ſhall, by the Property of the Curve, have $a^2 : b^2 :: ax - x^2$ (BS × AS) $: y^2$ (RS²), and therefore $b^2 \times \overline{ax - x^2} = a^2 y^2$: Whence $b^2 \times \overline{a\dot{x} - 2x\dot{x}} = 2a^2 y\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2a^2 y}{b^2 \times \overline{a - 2x}}$; and conſequently the Sub-tangent

* Art. 49. ST $\left(\frac{y\dot{x}}{\dot{y}}\right)$* $= \frac{2a^2 y^2}{b^2 \times \overline{a - 2x}} = \frac{a^2 y^2}{b^2 \times \overline{\frac{1}{2}a - x}} = \frac{b^2 \times \overline{ax - x^2}}{b^2 \times \overline{\frac{1}{2}a - x}} =$

$ax -$

$\frac{ax-x^2}{\frac{1}{2}a-x}$. Whence the Point T being given, through which the Tangent muſt paſs, the Tangent itſelf may be drawn.

But if you would derive an Expreſſion for the Sub-tangent, in any other kind of Ellipſes (beſides the conical) let the Equation $\overline{a-x}\rceil^{m}\times x^{n}=\frac{c}{a}\times y^{m+n}$, exhibiting the Nature of all Kinds of Ellipſes, be aſſumed: Then, by taking the Fluxion thereof, you will have $-m\dot{x}\times\overline{a-x}\rceil^{m-1}\times x^{n}+n\dot{x}x^{n-1}\times\overline{a-x}\rceil^{m}$ $=\frac{c}{a}\times\overline{m+n}\times y^{m+n-1}\dot{y}$; and therefore $\frac{y\dot{x}}{\dot{y}}=$

$$\frac{\frac{c}{a}\times\overline{m+n}\times y^{m+n}}{-m\times\overline{a-x}\rceil^{m-1}\times x^{n}+nx^{n-1}\times\overline{a-x}\rceil^{m}}$$

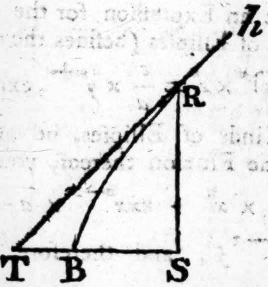
$=\frac{\overline{m+n}\times\overline{a-x}\rceil^{m}\times x^{n}}{-mx^{n}\times\overline{a-x}\rceil^{m-1}+nx^{n-1}\times\overline{a-x}\rceil^{m}}$ (becauſe $\frac{c}{a}\times$ $y^{m+n}=\overline{a-x}\rceil^{m}\times x^{n}$) $=\frac{\overline{m+n}\times\overline{a-x}\times x}{-mx+n\times\overline{a-x}}=$ $\frac{\overline{m+n}\times\overline{ax-x^2}}{na-\overline{n+m}\times x}$; which is the Sub-tangent required.

EXAMPLE V.

54. *To draw a Tangent, to any given Point* R, *in a given Hyperbola* BR*b*.

If a and c be put to denote the two principal Diameters of the Hyperbola, the Equation of the Curve will be $c^2\times\overline{ax+x^2}=a^2y^2$: From whence we have $c^2\times$

$\overline{a\dot{x}+2x\dot{x}}=2a^2y\dot{y}$, ∴ $\frac{\dot{x}}{\dot{y}}=\frac{a^2y}{c^2\times\overline{\frac{1}{2}a+x}}$, and conſequently $\frac{y\dot{x}}{\dot{y}}=\frac{a^2y^2}{c^2\times\overline{\frac{1}{2}a+x}}=\frac{c^2\times\overline{ax+x^2}}{c^2\times\overline{\frac{1}{2}a+x}}=\frac{ax+x^2}{\frac{1}{2}a+x}=ST$. Whence BT (ST — BS) $=\frac{\frac{1}{2}ax}{\frac{1}{2}a+x}$ is alſo known; and therefore the Point T being given the Tangent RT may be drawn.



The Manner of drawing Tangents to all Sorts of Hyperbolas, *univerſally*, will be the ſame as in the Ellipſes, the Equations of the two Kinds of Curves differing in Nothing but their Signs.

EXAMPLE VI.

55. *Let the propoſed Curve be that whoſe Equation is* $ax^2+xy^2+x^3-y^3=0$.

Then we ſhall have $2ax\dot{x}+y^2\dot{x}+2xy\dot{y}+3x^2\dot{x}-3y^2\dot{y}=0$; therefore $2ax\dot{x}+y^2\dot{x}+3x^2\dot{x}=3y^2\dot{y}-2xy\dot{y}$, $\frac{\dot{x}}{\dot{y}}=\frac{3y^2-2xy}{2ax+y^2+3x^2}$, and conſequently $\frac{y\dot{x}}{\dot{y}}=\frac{3y^3-2xy^2}{2ax+y^2+3x^2}$*.

*Art. 48 and 49.

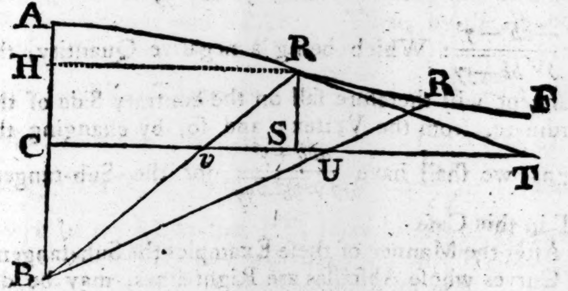
EXAMPLE VII.

56. *Let the given Curve be the Ciſſoid of* Diocles, *whoſe Equation is* $y^2 = \frac{x^3}{a-x}$.

Here we have $2y\dot{y} = \frac{3x^2\dot{x}\times\overline{a-x}+\dot{x}x^3}{\overline{a-x}|^2} = \frac{3ax^2\dot{x}-2x^3\dot{x}}{\overline{a-x}|^2}$; Whence $\frac{\dot{x}}{\dot{y}} = \frac{2y\times\overline{a-x}|^2}{3ax^2-2x^3}$, and conſequently the Subtangent $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2y^2\times\overline{a-x}|^2}{3ax^2-2x^3} = \frac{2x^3}{a-x}\times\frac{\overline{a-x}|^2}{3ax^2-2x^3} = \frac{2x\times\overline{a-x}}{3a-2x}$.

EXAMPLE VIII.

57. *Let the Conchoid of* Nicomedes *be propoſed*; whereof the Nature is ſuch, that, if from a Point B, called



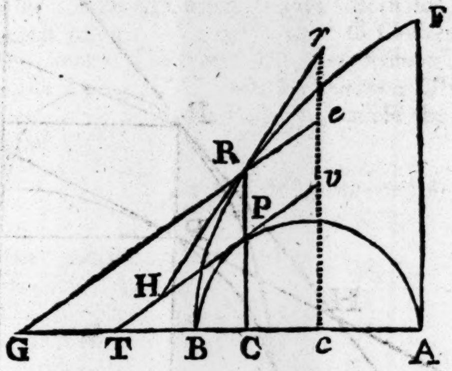
the Pole, any Number of Right-lines, BA, BR, BR, *&c.* be drawn, the Parts of thoſe Lines CA, *v*R, UR, *&c.* intercepted by the Curve and its Axis CT, ſhall be, all, equal to each other.

In this Caſe (ſuppoſing AB and RS perpendicular, and RH parallel, to CT; and putting BC $= a$, Rv (AC) $= b$, CS$= x$, and RS $= y$) we have, *per ſim. Triang.* $a+y$ (BH) : x (RH) :: y (RS) : $\frac{xy}{a+y} =$ Sv: But Sv ($\sqrt{\mathrm{R}v^2 - \mathrm{RS}^2}$) is alſo $= \sqrt{b^2-y^2}$; therefore $\frac{xy}{a+y} = \sqrt{b^2-y^2}$, or $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-y^2}$ is the general Equation of the Curve; which, in Fluxions, gives $2x^2y\dot{y} + 2y^2x\dot{x} = 2\dot{y} \times \overline{a+y} \times \overline{b^2-y^2} - 2y\dot{y} \times \overline{a+y}|^2 = 2\dot{y} \times \overline{a+y} \times \overline{b^2-ay-2y^2}$; and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times \overline{b^2-ay-2y^2} - x^2y}{xy^2}$, conſequently $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times y \times \overline{b^2-ay-2y^2} - x^2y^2}{y \times xy} = \frac{\overline{a+y} \times y \times \overline{b^2-ay-2y^2} - \overline{a+y}|^2 \times \overline{b^2-y^2}}{y \times \overline{a+y} \times \sqrt{b^2-y^2}}$ (becauſe $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-yy}$) $= \frac{b^2y - ayy - 2y^3 - abb + ayy - bby + y^3}{y\sqrt{bb-yy}} = \frac{-ab^2-y^3}{y\sqrt{bb-yy}}$: Which being a negative Quantity, the Tangent will therefore fall on the contrary Side of the Ordinate, from the Vertex; and ſo, by changing the Signs we ſhall have $\frac{abb+y^3}{y\sqrt{bb-yy}}$ for the Sub-tangent ST in this Caſe.

After the Manner of theſe Examples the Sub-tangent, in Curves whoſe Abſciſſas are Right-lines, may be determined: But if the Abſciſſa, or Line terminating the Ordinate, on the lower Part, be another Curve, then the Tangent may be drawn as in the following

EXAMPLE IX.

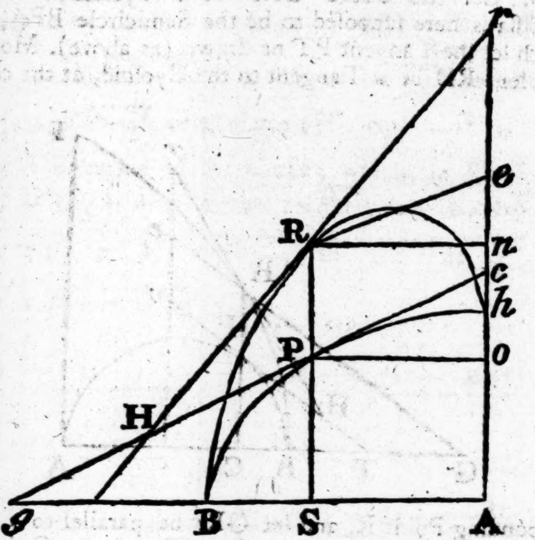
58. Let the Curve BRF be a Cycloid; whoſe Abſciſſa is here ſuppoſed to be the Semicircle BPA, to which let the Tangent PT be drawn (as above). Moreover let *r*RH be a Tangent to the Cycloid, at the correſponding Point R, and let GR*e* be parallel to TP*v*; putting the Arch (or Abſciſſa) BP$=z$, its Ordinate PR$=y$, AF$=b$, and BPA$=c$: Then, by the Property of the Curve, we ſhall have c (BPA) : b (AF) :: z (BP) : y (PR): Therefore $y=\frac{bz}{c}$, and $\dot{y}=\frac{b\dot{z}}{c}=re$: But, by ſimilar Triangles, re ($\dot{y}$) : Re ($=$ P$v=\dot{z}$) :: PR (y) : PH $=\frac{y\dot{z}}{\dot{y}}=z$ (becauſe $y=\frac{bz}{c}$). Therefore, if in the Right-line PT, there be taken PH, equal to the Arch PB, you will have a Point H, through which the Tangent of the Cycloid muſt paſs.



EXAMPLE X.

59. Let BP*b* be a Curve of any Kind, to which the Method of drawing the Tangent *c*P*g* is known; let BR*b*

BR*b* be another Curve of ſuch a Nature, that the Ordinate PR *(y)* ſhall always be a Mean-proportional between



BS *(x)* and AS *(a—x)* ſuppoſing RPS perpendicular to AB: Put P*o* $= \dot{x}$, SP $= v$, *oc* $= \dot{v}$ *, and *er* $= \dot{y}$: Then, *(as above)* *er* $(\dot{y})$: R*o* (= P*c* $= \sqrt{\dot{x}^2 + \dot{v}^2}$) :: RP *(y)* : PH $= \frac{y\sqrt{\dot{x}^2 + \dot{v}^2}}{\dot{y}}$: But, by the Equation of the Curve $y^2 = ax - xx$; whence $2y\dot{y} = a\dot{x} - 2x\dot{x}$, and $\frac{y}{\dot{y}} = \frac{2ax - 2x^2}{a\dot{x} - 2x\dot{x}}$, and therefore PH $=$ $\frac{2ax - 2x^2 \times \sqrt{\dot{x}^2 + \dot{v}^2}}{a\dot{x} - 2x\dot{x}}$: Which will be expreſſed independent of Fluxions, when the Property of the Curve BP*h*, or the Relation of *x* and *v* is given: Thus, let BP*h* be the common Parabola, and AB its *Latus Rectum*;

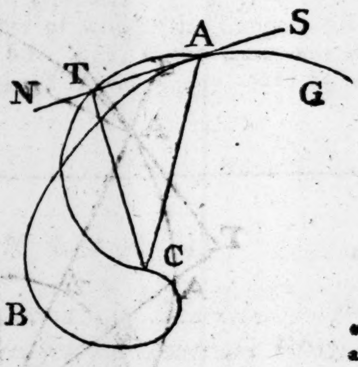
*Art. 48 and 49.

tum; then v being $= \sqrt{ax}$, $\dot{v}$ will be $= \frac{a\dot{x}}{2\sqrt{ax}}$, $\dot{x}^2 + \dot{v}^2 = \dot{x}^2 + \frac{a\dot{x}\dot{x}}{4x} = \frac{\dot{x}\dot{x} \times \overline{4x+a}}{4x}$; and therefore PH $\left(\frac{\overline{2ax - 2xx} \times \sqrt{\dot{x}^2 + \dot{v}^2}}{a\dot{x} - 2x\dot{x}}\right) = \frac{\overline{a-x} \times \sqrt{4x^2 + ax}}{a - 2x}$.

Thus far relates to Curves whoſe Ordinates are parallel to each other: We come now to Curves of the ſpiral Kind, whoſe Ordinates all iſſue from a Point: Such as the Spiral BAG, whoſe Ordinates CB, CA, CG, are all referred to the Point C, called the Center of the Spiral.

ILLUSTRATION.

60. Let SAN be a Tangent to the Spiral at any Point A, alſo let CT be perpendicular thereto, and let the Arch CBA (conſidered as variable by the Motion of A towards G) be denoted by z, and the Ordinate CA by y.



Then $\dot{z} : \dot{y} :: \text{AC}\ (y) : \text{AT} = \frac{y\dot{y}}{\dot{z}}$.*

Hence, if upon CA, as a Diameter, a Semi-circle be deſcribed, and in it, from A, a Right-line AT equal to $\frac{y\dot{y}}{\dot{z}}$ be inſcribed, that Right-line will be a Tangent to the Spiral at the Point A.

EXAMPLE I.

61. Let the Nature of the Curve CBA be ſuch that the Arch CBA may be, always, to its cor-

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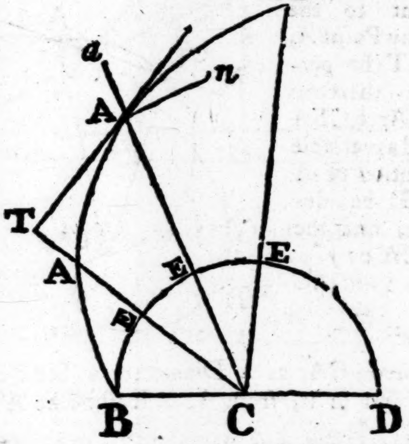
* Art. 5 and 35.

reſponding Ordinate CA in a conſtant Ratio; namely as a to b: Then, becauſe $z : y :: a : b$, we have $z = \frac{ay}{b}$, $\dot{z} = \frac{a\dot{y}}{b}$, and conſequently AT $\left(\frac{y\dot{y}}{\dot{z}}\right) = \frac{by}{a} = \frac{b}{a} \times$ AC: Therefore, AC and AT being in a conſtant Ratio, the Angle CAT muſt alſo be invariable. Which is a known Property of the logarithmic Spiral.

EXAMPLE II.

62. Let BAA be the Spiral of *Archimedes*; whoſe Nature is ſuch that the Part EA of the generating Ordinate, intercepted by the Spiral and a Circle BED deſcribed about the ſame Center C, is always in a conſtant Ratio to the correſponding Arch BE of that Circle.



Suppoſe An perpendicular to AC, &c.

Put BC $= c$, CA $= y$, and let the given Ratio of AE to BE, be that of b to c: Then $b : c :: y - c$ (AE) : $\frac{cy - cc}{b} =$ BE: whoſe Fluxion therefore is $= \frac{c\dot{y}}{b}$. Now if the Right-line CEA*a* be ſuppoſed to revolve about the Center C, the angular Celerity of the generating Point A, in the perpendicular Direction A*n*, will be to that of E as AC to EC; therefore as the latter of theſe Celerities is expreſſed by $\frac{c\dot{y}}{b}$*, the former will be expreſſed by $\frac{y}{c} \times \frac{c\dot{y}}{b}$, or $\frac{y\dot{y}}{b}$: Which is to ($\dot{y}$) the Celerity of A, in the Direction A*a*, as $\frac{y}{b}$ to Unity, or as y to b. Therefore CT and AT are in the ſame Ratio, (by *Art.* 35) and conſequently AC : CT :: $\sqrt{yy + bb}$: y; and AC : AT :: $\sqrt{yy + bb} : b$; whence CT and AT are given equal to $\frac{y^2}{\sqrt{yy + bb}}$, and $\frac{by}{\sqrt{yy + bb}}$ reſpectively. From either of which (the Tangent AT) may be drawn by *Art.* 60. And, in the ſame manner may the Poſition of the Tangent of any other Spiral be determined.

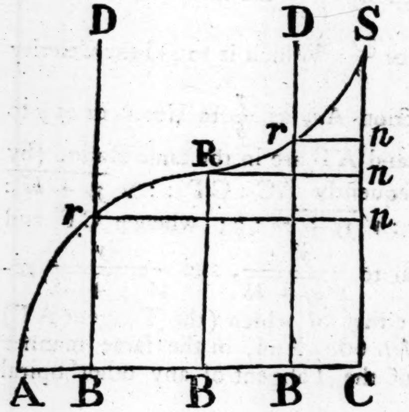
*Art. 5.

SECTION IV.

Of the Uſe of Fluxions in determining the Points of Retrogreſſion, or contrary Flexure in Curves.

63. WHEN a Curve ARS is, in one Part AR concave, and in the other Part RS convex, towards its Axis AC, the Point R limiting the two Parts is called a Point of Retrogreſſion, or contrary Flexure. The manner of determining which will appear from the following

ILLUSTRATION.



Suppoſe a Right-line BD to be carried along uniformly, parallel to itſelf, from A towards C; and let the Point *r* ſo move in that Line, at the ſame time, as to trace out, or deſcribe, the given Curve-line ARS. Then (*by Art.* 48.) while the Celerity of the Point *r*, in the Line BD, decreaſes, the Curve will be concave to its Axis AC; but when it increaſes, convex to the ſame: Therefore, as any Quantity is a *Minimum* at the End of its Decreaſe and the Beginning of its In-
*Art.22. creaſe *, it follows that the ſaid Celerity, at the Point of Inflexion R, muſt be a *Minimum*: Whence, if the
†Art. 5. Fluxion of the Ordinate B*r*, expreſſing that Celerity †, be (as uſual) denoted by $\dot{y}$; then will $\ddot{y}$ (the Fluxion
‡Art.22. of $\dot{y}$) be equal to Nothing in that Circumſtance ‡.

So far relates to Curves which are, in the former Part concave, and in the latter convex, to their Axes: But if (on the contrary) the Celerity of *r* firſt increaſes, and then decreaſes, that Celerity, at the required Point, between the Increaſe and Decreaſe, will be a *Maximum*; and therefore its Fluxion (or $\ddot{y}$) is *likewiſe* equal to
§Art.22. Nothing in this Caſe §.

Furthermore, if CS (perpendicular to AC) be now conſidered as an Axis, and the Abſciſſa S*n* (or its Complement B$r = y$) be ſuppoſed to flow uniformly, (as AB was ſuppoſed before); then, by the ſame Argument, the ſecond Fluxion ($-\ddot{x}$) of the Ordinate *nr* (or its Complement AB $= x$) will be equal to Nothing. Hence it is evident that, at the Point of contrary Flexure, the ſecond Fluxion of the Ordinate will become equal to Nothing, if the Abſciſſa be made to flow uniformly; and *vice verſâ.*

EXAMPLE I.

64. Let the Nature of the Curve ARS (*ſee the preceding Figure*) be defined by the Equation $ay = a^{\frac{3}{2}}x^{\frac{1}{2}} + xx$ (the Abſciſſa AB and the Ordinate B*r* being, as uſual, repreſented by x and y reſpectively). Then $\dot{y}$, expreſſing the Celerity of the Point *r*, in the Line BD, will be equal to $\frac{\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} + 2x\dot{x}}{a}$: Whoſe Fluxion, or that of $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} + 2x$ (becauſe a and $\dot{x}$ are conſtant) muſt be equal to Nothing *; that is, $-\frac{1}{4}a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x} + 2\dot{x} = 0$: Whence $a^{\frac{3}{2}}x^{-\frac{3}{2}} = 8$, $a^{\frac{3}{2}} = 8x^{\frac{3}{2}}$, $64x^3 = a^3$, and $x = \frac{1}{4}a = \text{AB}$; therefore BR $\left(= \frac{a^{\frac{3}{2}}x^{\frac{1}{2}} + xx}{a}\right) = \frac{9}{16}a$: From which the Poſition of the Point R is given. * Art. 63.

EXAMPLE II.

65. Let the Nature of the propoſed Curve be defined by the Equation $ayy - aax - x^3 = 0$.

Then, by taking the firſt and ſecond Fluxions thereof (ſuppoſing $\dot{x}$ conſtant) we ſhall alſo have $2ay\dot{y} - aa\dot{x} - 3x^2\dot{x} = 0$, and $2a\dot{y}^2 + 2ay\ddot{y} - 6x\dot{x}\dot{x} = 0$; whereof the latter, when $\ddot{y}$ is $= 0$, becomes $2a\dot{y}^2 - 6x\dot{x}^2 = 0$, and therefore $\dot{y}^2 = \frac{3x\dot{x}^2}{a}$: But, by the former $\dot{y} = \frac{a^2\dot{x} + 3x^2\dot{x}}{2ay}$; whence $\frac{3x\dot{x}^2}{a} = \frac{\overline{a^2\dot{x} + 3x\dot{x}}|^2}{\overline{2ay}|^2}$, and conſequently $12axy^2$

$= \overline{a^2 + 3x^2}|^2$; but, by the given Equation, $12axy^2 = 12a^2x^2 + 12x^4$, therefore $12a^2x^2 + 12x^4 = \overline{a^2 + 3x^2}|^2$, or $3x^4 + 6a^2x^2 - a^4 = 0$: Whence x will be found $= a\sqrt{:\sqrt{\tfrac{4}{3}} - 1}$.

Otherwiſe.

Since $ay^2 = a^2x + x^3$, we have $y = \frac{\overline{a^2x + x^3}|^{\frac{1}{2}}}{\sqrt{a}}$, and therefore $\dot{y} = \frac{\overline{\tfrac{1}{2}a^2\dot{x} + \tfrac{3}{2}x^2\dot{x}} \times \overline{a^2x + x^3}|^{-\frac{1}{2}}}{\sqrt{a}}$: Whoſe Fluxion, or that of $\overline{a^2 + 3x^2} \times \overline{a^2x + x^3}|^{-\frac{1}{2}}$ (becauſe $\dot{x}$ is conſtant) being put $= 0$, we get $6x \times \overline{a^2x + x^3}|^{-\frac{1}{2}} + \overline{a^2 + 3x^2} \times \overline{-\tfrac{1}{2}a^2 - \tfrac{3}{2}x^2} \times \overline{a^2x + x^3}|^{-\frac{3}{2}} = 0$, or $6x \times \overline{a^2x + x^3} + \overline{a^2 + 3x^2} \times -\frac{\overline{a^2 + 3x^2}}{2}$: Whence $3x^4 + 6a^2x^2 - a^4 = 0$, and $x = a\sqrt{:\sqrt{\tfrac{4}{3}} - 1}$, the ſame as before.

EXAMPLE III.

66. Let the propoſed Curve be the Conchoid of *Nicomedes*, whereof the Equation is $x^2y^2 = \overline{a + y}|^2 \times$
Art. 57. $\overline{b^2 - y^2}$ *, or $x^2 = \frac{\overline{a + y}|^2 \times \overline{b^2 - y^2}}{y^2}$.

Here we have $x\dot{x}=\dfrac{\overline{\dot{y}\times\overline{a+y}+\overline{b^2-y^2}-y\dot{y}\times\overline{a+y}|^2\times y^2}}{y^4}$ $\dfrac{-y\dot{y}\times\overline{a+y}|^2\times\overline{b^2-y^2}}{y^4}=-\dfrac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y}=$ $\overline{\dfrac{-a^2b^2}{y^3}-\dfrac{ab^2}{y^2}-a-y}\times\dot{y}$: Whence, making $\dot{y}$ invariable, we alſo have $\dot{x}^2+x\ddot{x}=\overline{\dfrac{3a^2b^2}{y^4}+\dfrac{2ab^2}{y^3}-1}\times\dot{y}^2$: Which, becauſe $\ddot{x}$ is $=0$ *, will be $\dot{x}^2=\overline{\dfrac{3a^2b^2}{y^4}+\dfrac{2ab^2}{y^3}-1}$ *Art.63. $\times\dot{y}^2=\dfrac{3a^2b^2+2ab^2y-y^4}{y^4}\times\dot{y}^2$. But ſince, by the former Equation, $x\dot{x}=-\dfrac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y}$, we likewiſe get $\dot{x}^2=\dfrac{\overline{a+y}|^2\times\overline{ab^2+y^3}|^2}{x^2y^6}\times\dot{y}^2$, and conſequently $\overline{3a^2b^2+2ab^2y-y^4}\times x^2y^2=\overline{a+y}|^2\times\overline{ab^2+y^3}|^2$: But, by the Equation of the Curve x^2y^2 is $=\overline{a+y}|^2\times\overline{b^2-y^2}$; therefore $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{a+y}|^2\times\overline{b^2-y^2}=\overline{a+y}|^2$ $\times\overline{ab^2+y^3}|^2$, and $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{b^2-y^2}=\overline{ab^2+y^3}|^2$; whence $y^4+4ay^3+3a^2y^2-2ab^2y-2a^2b^2=0$; which divided by $y+a$, gives $y^3+3ay^2-2ab^2=0$; from whence y may be determined. But if $b=a$, the Equation will become more ſimple by dividing again by $y+a$; in which Caſe we get $y^2+2ay-2a^2=0$, and conſequently $y=a\sqrt{3}-a$.

EXAMPLE IV.

67. Let $a^4y=180a^3x^2-110a^2x^3+30ax^4-3x^5$.

Then will $a^4\dot{y}=360a^3x\dot{x}-330a^2x^2\dot{x}+120ax^3\dot{x}-15x^4\dot{x}$;

And $a^4\ddot{y} = 360a^3\dot{x}^2 - 660a^3x\dot{x}^2 + 360ax^2\dot{x}^2 - 60x^3\dot{x}^2$

* Art. 63. Therefore, $6a^3 - 11a^2x + 6ax^2 - x^3 = 0$ *:

Which being diviſible by any one of the three Quantities $a - x$, $2a - x$, or $3a - x$, the Root x muſt therefore have three Values, a, $2a$, and $3a$, and conſequently the Curve, defined by the given Equation, as many Points of contrary Flexure.

But, if you would know whether the Part of the Curve lying between any two adjacent Points, thus found, be convex or concave towards the Axis; ſee whether the Value of the Expreſſion for the ſecond Fluxion of the Ordinate, between the two correſponding Roots, be poſitive or negative: For, in the former Caſe, the Curve is convex, and in the latter concave †, (provided the *whole* Curve lies on the ſame Side the Axis). Thus, in the Example before us; becauſe the ſecond Fluxion of the Ordinate is always as $6a^3 - 11aax + 6axx - x^3$ $(= \overline{a-x} \times \overline{2a-x} \times \overline{3a-x})$ and it appears that the Value of this Expreſſion, while x is leſs than the firſt Root a, will be poſitive; the Curve, therefore, at the Beginning, will be convex to its Axis: But when x becomes greater than a, the ſaid Expreſſion being negative, the Curve will then be concave, and ſo continue 'till x is equal to the ſecond Root $2a$; after which the Fluxion again becoming affirmative, the Curve will accordingly be convex till $x = 3a$; beyond which Limit the Curvature continually tends the ſame Way. † Art. 5 and 48.

But it will be proper to obſerve, that there are Caſes where the ſecond Fluxion of the Ordinate may become equal to Nothing, without either changing its Value from poſitive to negative, or the contrary, (ſimilar to thoſe already taken Notice of in *Sect.* II. *p.* 45 and 46.) which Caſes always happen when the Equation admits of an even Number of equal Roots: And then the Point found as above is not a Point of Inflexion, becauſe the Curvature on either Side of it tends the ſame Way.

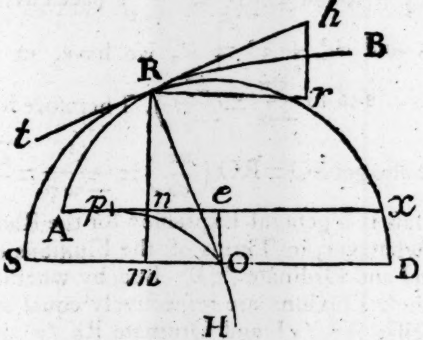
SECTION V.

The Uſe of Fluxions in determining the Radii of Curvature, and the Evolutes of Curves.

68. A Curve pOH is ſaid to be the Evolute of another Curve ARB, when it is of ſuch a Nature, that a Thread ROH, coinciding therewith (or wrapped upon the ſame) being unwound or diſengaged from it, by a Power acting at the End R, ſhall, by that End (the Thread continuing tight) deſcribe the given Curve ARB.

ILLUSTRATION.

From the Point O, where the Right-line RO (called the Radius of Curvature) touches the Evolute pOH,



let the Semi-circle SRD be deſcribed; which Semi-circle, having the ſame Radius with the given Curve, at R, will conſequently have the ſame Degree of Curvature.——But the Curvature in two Curves is the ſame, when, the Fluxions of their Abſciſſas being the ſame, both the Firſt, and Second Fluxions of their correſponding Ordinates R*n* and R*m* are reſpectively equal to each other: For, the Firſt Fluxions being equal, the two Curves will have, at the common Point R, one and the ſame Tangent *t*R*h* *: And, if the Second Fluxions be likewiſe equal, the Curvature, or Deflection from that Tangent, will alſo be the ſame in both; becauſe theſe laſt expreſs the Increaſe or Decreaſe of Motion in the Direction of the Ordinate †, upon which the Curvature intirely depends ‡.

* Art. 48.
† Art. 19.
‡ Art. 48.

This being premiſed, let the Abſciſſa S*m* of the Semicircle (conſidered as variable) be put $= w$, its Ordinate R*m* $= v$, R*r* $= \dot{w}$, *rh* $= \dot{v}$, and R*h* $= \dot{z}$: Then, R*h* being a Tangent to the Circle at R ‖, the Triangles R*hr* and RO*m* will be equiangular, and therefore $\dot{w}$ (R*r*) : $\dot{z}$ (RA) :: v (R*m*) : RO $= \frac{v\dot{z}}{\dot{w}}$; which, becauſe the Radius of every Circle is a conſtant Quantity, muſt be invariable, and conſequently its Fluxion $\frac{\dot{v}\dot{z}+v\ddot{z}}{\dot{w}} = 0$: Whence v is found $= \frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$ (becauſe, $\dot{w}$ being conſtant, and $\dot{w}^2 + \dot{v}^2 = \dot{z}^2$, we have, in Fluxions $2\dot{v}\ddot{v} = 2\dot{z}\ddot{z}$, and ſo $\frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$). Therefore ſince v is $= \frac{\dot{z}^2}{-\ddot{v}}$, we alſo get SO=RO $\left(\frac{v\dot{z}}{\dot{w}}\right) = \frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\overline{\dot{v}^2+\dot{w}^2}|^{\frac{1}{2}}}{-\dot{w}\ddot{v}}$: Which laſt is a general Expreſſion for the Radius of any Circle, whatever, in Terms of the Fluxions of its Abſciſſa *(w)* and Ordinate *(v)*. But, by what is premiſed above, theſe Fluxions are reſpectively equal to thoſe of the Abſciſſa A*n* *(x)* and Ordinate R*n* *(y)* of the propoſed Curve ARB. Therefore, by writing $\dot{x}$, $\dot{y}$, and $\ddot{y}$, inſtead of $\dot{w}$, $\dot{v}$, and $\ddot{v}$, we have $\frac{\overline{\dot{y}^2+\dot{x}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}} \left(= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)$ for the general Value of the Radius of Curvature, RO.

‖ Art. 48.

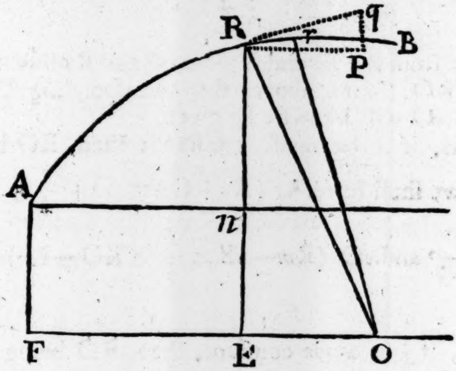
The ſame otherwiſe.

If the Radius of the Circle be put $= R$, and every Thing elſe be ſuppoſed as above; then (by the Property of the Circle) we ſhall have v^2 (Rm^2) $= 2\,Rw - w^2$ ($Sm \times Dm$) : Whence, in Fluxions (making $\dot{w}$ conſtant) we get $2v\dot{v} = 2R\dot{w} - 2w\dot{w}$, and $2\dot{v}^2 + 2v\ddot{v} = -2\dot{w}^2$: From the laſt of which Equations v is found $= \dfrac{\dot{v}^2 + \dot{w}^2}{-\ddot{v}}$ $= \dfrac{\dot{z}^2}{-\ddot{v}}$; and conſequently RO $\left(\dfrac{v\dot{z}}{\dot{w}}\right) = \dfrac{\dot{z}^3}{-\dot{w}\ddot{v}} = \dfrac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *the ſame as before.*

Otherwiſe without the Circle.

Let RO and rO be two Rays perpendicular to the Curve, indefinitely near to each other; and from their Interſection O, let OF be drawn parallel to An, cutting Rn and AF (parallel to Rn) in E and F.

Therefore, ſuppoſing RE$=v$, A$n$$=x$, R$n$$=y$, &c. (as before) we ſhall have, by ſimilar Triangles, as RP



($\dot{x}$) : Pq ($\dot{y}$) :: RE (v) : EO $= \dfrac{v\dot{y}}{\dot{x}}$; and conſequently FO (A$n$+EO) $= x + \dfrac{v\dot{y}}{\dot{x}}$: Which Value (as well as that of AF) continuing the ſame whether we regard the Radius RO, or the Radius *r*O, its Fluxion muſt therefore be equal to Nothing; that is, $\dot{x} + \frac{\overline{\dot{v}\dot{y} + v\ddot{y}} \times \dot{x} - v\dot{y}\ddot{x}}{\dot{x}^2} = 0$; whence $v = \frac{\dot{x}^3 + \dot{x}\dot{v}\dot{y}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, and conſequently RO $\left(\frac{v\dot{z}}{\dot{x}}\right) = \frac{\dot{x}^2\dot{z} + \dot{v}\dot{y}\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{x}^2\dot{z} + \dot{y}^2\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{z}^3}{\dot{y}\ddot{x} - x\ddot{y}}$: Which, if $\dot{x}$ is ſuppoſed conſtant, or $\ddot{x} = 0$, will become $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *as above*.

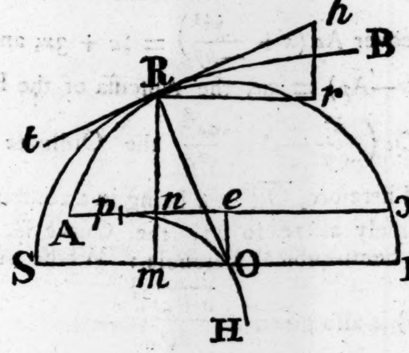
But if $\dot{y}$ be ſuppoſed conſtant, it will be $\frac{\dot{z}^3}{\dot{x}\ddot{y}}$. And, if $\dot{z}$ be conſtant, it will then be $\frac{\dot{z}\dot{y}}{\ddot{x}}$: For, ſince $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, by taking the Fluxion thereof, we have $2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0$; whence $\ddot{y} = -\frac{\dot{x}\ddot{x}}{\dot{y}}$; and therefore RO $\left(= \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}\right) = \frac{\dot{z}^3}{\dot{y}\ddot{x} + \frac{\dot{x}^2\ddot{x}}{\dot{y}}} = \frac{\dot{y}\dot{z}^3}{\overline{\dot{y}^2 + \dot{x}^2} \times \ddot{x}} = \frac{\dot{y}\dot{z}}{\ddot{x}}$, *as before.*

Now from the ſeveral Values of the Radius of Curvature RO, found above, the correſponding Values of A*e* and *e*O will likewiſe be given.

Thus, if $\dot{x}$ be made conſtant; then, RO being $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, we ſhall have A*e* (A*n* + O*m* = A*n* + $\frac{\dot{y}}{\dot{z}}$ × RO) $= x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}$, and *e*O (R*m* — *n*R = $\frac{\dot{x}}{\dot{z}}$ × RO — R*n*) $= \frac{\dot{z}^2}{-\ddot{y}} - y$.

But, if $\dot{y}$ be made conſtant, then, RO being $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, we ſhall have AE $= x + \frac{\dot{z}^2}{\ddot{x}}$, and *e*O $= \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x}} - y$.

Laſtly, if $\dot z$ be ſuppoſed conſtant; then RO being $= \frac{\dot y \dot z}{\ddot x}$, we ſhall have A*e* $= x + \frac{\dot y^2}{\ddot x}$, and *e*O $= \frac{\dot x \dot y}{\ddot x} - y$.



Which ſeveral Expreſſions will ſerve as ſo many general Theorems for determining the Quantity of Curvature, and the Evolutes of given Curves: But, before we proceed to Examples, it will be proper to obſerve, that the Right-line A*p*, denoting the Radius of Curvature at the Vertex A (to be found by making *x*, or *y*, $= 0$) muſt always be ſubtracted from RO and A*e*, to have the true Length of the Arch *p*O, and its correſponding Abſciſſa *pe*.

EXAMPLE I.

69. Let the given Curve ARB be the common Parabola, whoſe Equation is $y = a^{\frac{1}{2}} x^{\frac{1}{2}}$: Then will $\dot y = \frac{1}{2} a^{\frac{1}{2}} \dot x x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}} \dot x}{2 x^{\frac{1}{2}}}$, and (making $\dot x$ conſtant) $\ddot y = -\frac{1}{2} \times \frac{1}{2} a^{\frac{1}{2}} \dot x^2 x^{-\frac{3}{2}} = \frac{-a^{\frac{1}{2}} \dot x^2}{4 x^{\frac{3}{2}}}$: Whence $\dot z$ $(\sqrt{\dot x + \dot y^2}$ $= \frac{\dot x}{2} \sqrt{\frac{4x + a}{x}}$,

and the Radius of Curvature RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}}$: Which at the Vertex A, where $x=0$, will be $= \frac{1}{2}a =$ A*p*. Moreover A*e* $\left(x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right) = \frac{1}{2}a + 3x$, and therefore *pe* (A*e*—A*p*) $= 3x$, the Abſciſſa of the Evolute: Likewiſe O*e* $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right) = \frac{4x^{\frac{3}{2}}}{\sqrt{a}}$ the Ordinate of the Evolute. Therefore, $\overline{Oe}|^2 \times a$ being in a conſtant Ratio to $\overline{pe}|^3$, namely as 16 to 27, the Curve is, in this Caſe, the Semi-cubical Parabola: Whoſe Arch *p*O (RO—A*p*) is alſo given $= \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}} - \frac{1}{2}a$.

EXAMPLE II.

70. Let the Curve ARB denote a Parabola of any other Kind: Then, becauſe $y = ax^n$ is an Equation to all Kinds of Parabolas, we have $\dot{y} = nax^{n-1}\dot{x}$ and $\ddot{y} = n \times \overline{n-1} \times ax^{n-2}\dot{x}^2$: Therefore $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}) = \dot{x}\sqrt{1+n^2a^2x^{2n-2}}$, RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{1+n^2a^2x^{2n-2}}|^{\frac{3}{2}}}{-n\times\overline{n-1}\times ax^{n-2}}$, A*e* $\left(x + \frac{\dot{y}\dot{z}^2}{-\ddot{y}}\right) = x - \frac{x+n^2a^2x^{2n-1}}{n-1}$, O*e* $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right) = \frac{1 + \overline{2n-1} \times na^2x^{2n-2}}{-\overline{n-1} \times nax^{n-2}}$, and A*p* $= -\frac{n^2a^2 0^{2n-1}}{n-1}$: Which, if $n=\frac{1}{2}$, will become $= \frac{a^2}{2}$; but, if *n* be greater than $\frac{1}{2}$, it will be $= 0$; and, if *n* be leſs than $\frac{1}{2}$,

it will be infinite: Whence it appears, that the Radius of Curvature at the Vertex will be a finite Quantity in Curves whoſe firſt (or leaſt) Ordinates are in the Subduplicate Ratio of their Abſciſſas, and in all other Caſes, either Nothing, or Infinite.

EXAMPLE III.

71. Suppoſe the given Curve to be an Ellipſis; whoſe Equation (putting a and c for the two principal Diameters) is $a^2y^2 = c^2 \times \overline{ax - x^2}$.

Here, by taking the Firſt and Second Fluxions of the given Equation, we have $2a^2y\dot{y} = c^2\dot{x} \times \overline{a - 2x}$, and $2a^2y^2 + 2a^2y\ddot{y} = c^2\dot{x} \times -2\dot{x} = -2c^2\dot{x}^2$; whence $\dot{y} = \frac{c^2\dot{x} \times \overline{a - 2x}}{2a^2y}$, and $-\ddot{y} = \frac{a^2\dot{y}^2 + c^2\dot{x}^2}{a^2y}$: Which, by ſubſtituting the Values of y and $\dot{y}$, will become $\dot{y} = \frac{c\dot{x} \times \overline{a - 2x}}{2a\sqrt{ax - x^2}}$, and $-\ddot{y} = \frac{a^2c^2\dot{x}^2 \times \overline{a - 2x}\big|^2}{4a^2 \times \overline{ax - xx} \times ac\sqrt{ax - x^2}}$ $+ \frac{c\dot{x}^2}{a\sqrt{ax - x^2}} = \frac{c\dot{x}^2}{a} \times \frac{\overline{a - 2x}\big|^2 + 4 \times \overline{ax - x^2}}{4 \times \overline{ax - x^2}\sqrt{ax - x^2}} = \frac{ca\dot{x}^2}{4 \times \overline{ax - x^2}\big|^{\frac{3}{2}}}$:

Therefore $\dot{z}\ (\sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\frac{c^2\dot{x}^2 \times \overline{a - 2x}\big|^2}{4a^2 \times \overline{ax - x^2}} + \dot{x}^2}$ $= \frac{\dot{x}}{2a}\sqrt{\frac{c^2a^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}{ax - x^2}}$, and the Radius of Curvature $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a^2c^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}\big|^{\frac{3}{2}}}{2a^4c}$: Which when the Diameters a and c are equal, or the Ellipſis degenerates to a Circle, will be every where equal to $\frac{\overline{a^2c^4}\big|^{\frac{3}{2}}}{2a^4c}$, or $\frac{1}{2}a$; agreeable to the Definition of a Circle.

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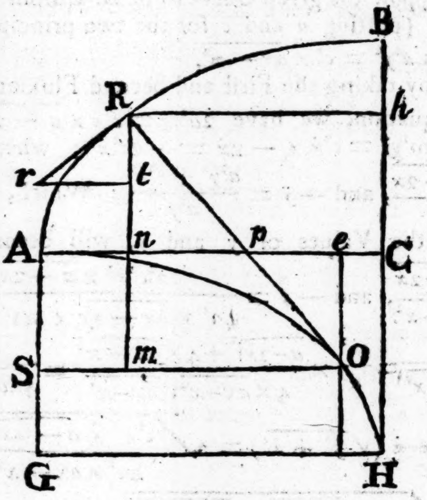
EXAMPLE IV.

72. *To find the Radius of Curvature, and the Evolute of the common Cycloid.*

Let ARB be the given Curve, and AOH its Evolute; alſo let Rh and OS be parallel to AC, and tO and Rm



perpendicular to AC; and put ARB ($=2$BC) $= a$, AR $= z$, A$n = x$, and R$n = y$: Then BR $= a - z$, Bh $= \frac{1}{2}a - y$; and, by the Property of the Curve, a^2 (AB²) : $\overline{a-z}|^2$ (BR²) : : $\frac{1}{2}a$ (BC) : $\frac{1}{2}a - y$ (Bh) whence $y = \frac{2az - z^2}{2a}$; therefore $\dot{y} = \frac{a\dot{z} - z\dot{z}}{a}$, $\dot{z}^2 - \dot{y}^2$ $(\dot{x}^2) = \frac{\overline{2az - z^2} \times \dot{z}^2}{a^2}$, and $\dot{x} = \frac{\dot{z}\sqrt{2az - z^2}}{a}$. Whence (making $\dot{z}$ conſtant) $\ddot{x} = \frac{\dot{z}^2 \times \overline{a - z}}{a\sqrt{2az - z^2}}$; from which we get RO, or AO $\left(=\frac{\dot{y}\dot{z}}{\ddot{x}}\right)$ * $=\sqrt{2az-z^2}$, and *e*O, *Art.68. or AS $\left(=\frac{\dot{y}\dot{x}}{\ddot{x}}-y\right)=\frac{2az-z^2}{2a}$; which, when $z=a$, or ROH coincides with BH, become AOH (BH) $=a$, and CH (AG) $=\frac{1}{2}a$. Hence, becauſe it appears, that, $\overline{\text{AH}}|^2$ (a^2) : AO^2 $(2az-z^2)$ :: AG $(\frac{1}{2}a)$: AS $\left(\frac{2az-z^2}{2a}\right)$ it follows that the Evolute AOH is alſo a Cycloid equal, and ſimilar, to the Involute ARB.

If the Evolute had been given, or ſuppoſed, a Cycloid, and the Involute required, the Proceſs would have been, more ſimple, as follows,

Let AH (2AG) $=a$, AO ($=$RO) $=z$, AS $=x$, SO $=y$, BR $=v$, B*b* $=w$, R*r* $=\dot{v}$, R*t* $=\dot{w}$, &c. Then it will be †, †Art.48.

$$\dot{y}:\dot{z}\ (::\text{O}m:\text{OR})::\text{R}t\ (\dot{w}):\text{R}r=\frac{\dot{w}\dot{z}}{\dot{y}}.$$

$$\dot{z}:\dot{y}::z\,(\text{RO}):\text{O}m=\frac{z\dot{y}}{\dot{z}},$$

$$\dot{z}:\dot{x}::z\,(\text{RO}):\text{R}m=\frac{z\dot{x}}{\dot{z}},$$

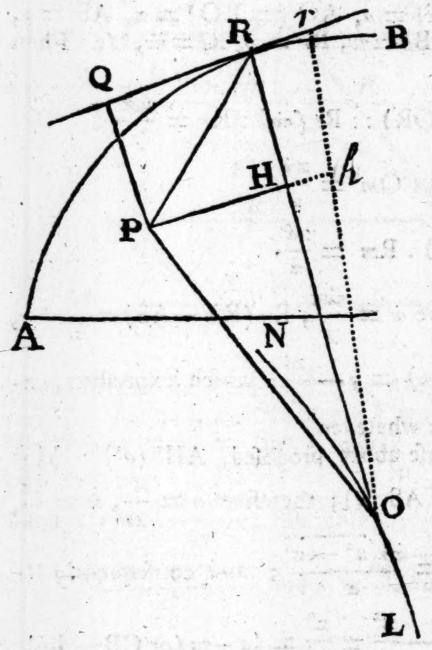
Whence we have $\dot{v}=\frac{\dot{w}\dot{z}}{\dot{y}}$, R*n* (R*m*$-$AS) $=\frac{z\dot{x}}{\dot{z}}-x$, and A*n* (OS$-$O*m*) $=y-\frac{z\dot{y}}{\dot{z}}$; which Expreſſions anſwer to any Curve whatever.

But, in the Caſe above propoſed, AH^2 (a^2) : AO^2 (z^2) :: AG $(\frac{1}{2}a)$: AS (x); therefore $x=\frac{z^2}{2a}$, $\dot{x}=\frac{z\dot{z}}{a}$, and $\dot{y}$ $(\sqrt{\dot{z}^2-\dot{x}^2})=\frac{\dot{z}\sqrt{a^2-z^2}}{a}$; and conſequently R*n* $\left(\frac{z\dot{x}}{\dot{z}}-x\right)=\frac{z^2}{a}-\frac{z^2}{2a}=\frac{z^2}{2a}=\frac{1}{2}a-w$ (or CB$-$B*b*):

Whence alſo $w = \frac{a^2 - z^2}{2a}$, and $\dot{v} \left(\frac{\dot{w}\dot{z}}{y}\right) = \frac{a\dot{w}}{\sqrt{a^2 - z^2}}$ $= \frac{a\dot{w}}{\sqrt{2aw}}$: Therefore it will be $\dot{v} : \dot{w}$ $(:: a : \sqrt{2aw})$ $:: \sqrt{\frac{1}{2}a} : \sqrt{w}$; that is, as R$r$: Rt :: $\sqrt{\text{BC}} : \sqrt{\text{B}h}$: Which is a known Property of the Cycloid.

Hitherto regard has been had to Curves where the Ordinates are parallel to each other: But when the Ordinates are all referred to a given Point, as in Spirals, *&c.* other Theorems will become neceſſary; and may be thus derived.

73. Let ARB be the propoſed Curve, P the Point, or Center, to which its Ordinates are referred, NOL the Evolute, and RO the Ray of Curvature at R: Moreover, let PH be perpendicular to RO; and, ſuppoſing the Ordinate PR (y) to become variable by the Motion of the Point R along the Curve, let the Fluxions of AR and PH (p), expreſſing the Celerities of the Points R and H in Directions perpendicular to RO *, be denoted by $\dot{z}$ and $\dot{p}$ reſpectively.



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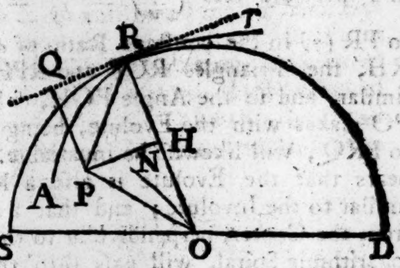
* Art. 5.

Therefore, the Celerities, of any two Points, in a Right-line revolving about a Center, being as the Diſtances from that Center, it follows that $\dot{p} : \dot{z} :: \text{OH} : \text{OR}$; whence by Diviſion (putting $\text{RH} = v$) we have $\dot{z} - \dot{p} : \dot{z} :: v\ (\text{RH}) : \text{RO} = \frac{v\dot{z}}{\dot{z} - \dot{p}} = \frac{vp\dot{z}}{p\dot{z} - p\dot{p}}$: But $p\dot{z} = y\dot{y}$ (*by Art.* 60.) and therefore $\text{RO} = \frac{vy\dot{y}}{y\dot{y} - p\dot{p}}$; which, becauſe $y^2 - p^2$ is $= v^2$ (and therefore $y\dot{y} - p\dot{p} = v\dot{v}$) will alſo be $= \frac{vy\dot{y}}{v\dot{v}} = \frac{y\dot{y}}{\dot{v}}$.

The ſame otherwiſe.

Let SRD be a Circle deſcribed about the Point O, as a Center, and ſuppoſe the Diſtance PR to be variable by the Motion of the Point R along the Arch of the Circle (inſtead of the Curve): Then, drawing OP, and putting $\text{OR} = r$, $\text{PR} = y$, &c. as before, we ſhall get OP^2 $(\text{OR}^2 + \text{PR}^2 - 2\text{OR} \times \text{RH}) = r^2 + y^2 - 2rv$; which (as well as r) being a conſtant Quantity, its Fluxion $2y\dot{y} - 2r\dot{v}$ muſt be equal to nothing; and therefore $r = \frac{y\dot{y}}{\dot{v}}$, the very ſame as above. Nor is it of any Conſequence whether $\dot{y}$ and $\dot{v}$ be here looked upon as reſpecting the Circle, or the Curve; ſince, at R, they muſt be the ſame in both Caſes; otherwiſe the Curvature could not be the ſame *. Now from the Value of RO thus found, which (corrected, when neceſſary) will alſo expreſs the Length of the Arch NO of the Evolute ‡, the Ordinate PO and the Tangent OH of the Evolute may



* Art. 68.

‡ Art. 68.

may be eaſily deduced. For OH (RO—RH) $= \frac{y\dot{y}}{\dot{v}} - v = \frac{p\dot{p}}{\dot{v}}$, and PO ($= \sqrt{OH^2 + PH^2} = \frac{p\sqrt{\dot{p}^2 + \dot{v}^2}}{\dot{v}}$); whence the Nature of the Evolute is known.

EXAMPLE I.

74. Let the given Curve AR be the logarithmic Spiral, whoſe Nature is ſuch, that the Angle PRQ (or RPH) which the Ordinate makes with the Curve is every where the ſame.

Then (denoting the Sine of that Angle by b, and the Radius of the Tables by a) we have RH (v) $= \frac{by}{a}$, and therefore RO $\left(\frac{y\dot{y}}{\dot{v}}\right) = \frac{ay\dot{y}}{b\dot{y}} = \frac{ay}{b}$; which being to PR ($y$) in the conſtant Ratio of a to b, or of PR to RH, the Triangles ROP and RPH muſt therefore be ſimilar, and ſo the Angle POH, which the Ordinate PO makes with the Evolute, being every where equal to PRQ, will likewiſe be invariable. Whence it appears that the Evolute is alſo a logarithmic Spiral, ſimilar to the Involute; and that a Right-line drawn from the Center, perpendicular to the Ordinate, of any logarithmic Spiral, will paſs thro' the Center of Curvature.

EXAMPLE II.

75. Let the Curve propoſed be the Spiral of *Archimedes*; where we have $p = \frac{by}{\sqrt{y^2 + b^2}}$, and $v = \frac{y^2}{\sqrt{y^2 + b^2}}$ (ſee *Art.* 62.) Therefore $\dot{v} = 2y\dot{y} \times \overline{y^2 + b^2}\,|^{-\frac{1}{2}} + y\dot{y} \times$

$-\frac{1}{2}$

$$-\tfrac{1}{2} \times 2y\dot{y} \times \overline{y^2+b^2}\big|^{-\frac{3}{2}} \pm \frac{2y\dot{y}}{\overline{y^2+b^2}\big|^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+b^2}\big|^{\frac{3}{2}}} =$$

$$\frac{2y\dot{y} \times \overline{y^2+b^2} - y^3\dot{y}}{\overline{y^2+b^2}\big|^{\frac{3}{2}}} = \frac{y^3\dot{y}+2b^2y\dot{y}}{\overline{y^2+b^2}\big|^{\frac{3}{2}}}$$; whence the Radius of * Curvature $\frac{y\dot{y}}{\dot{v}}$ is here $= \frac{\overline{yy+bb}\big|^{\frac{3}{2}}}{y^2+2b^2}$; which being $= \frac{b}{2}$, when $y = 0$, the Arch of the Evolute †, reckoned from the Vertex, is therefore $= \frac{\overline{yy+bb}\big|^{\frac{3}{2}}}{y^2+2b^2} - \frac{b}{2}$. *Art. 73. †Art. 68

After the very ſame Manner you may proceed in other Caſes: But if the Value of $\dot{v}$ (or $\frac{y\dot{y}}{\dot{v}}$) changes, in any Caſe, from Poſitive to Negative, the Radius of Curvature (RO) after becoming infinite, will fall on the other Side of the Tangent, and the correſponding Point of the Curve, when $\dot{v} = 0$, will be a Point of *Contrary-Flexure*. Whence it may be obſerved that the Point of Inflection, in a Curve whoſe Ordinates are referred to a Center, may be found by making the Fluxion of the Perpendicular, drawn from the Center to the Tangent, equal to Nothing, which Caſe is not taken Notice of in the preceding Section.

SECTION VI.

Of the Inverſe Method, or the Manner of determining the Fluents of given Fluxions.

76. IN the *Inverſe Method*, which teaches the Manner of finding the reſpective flowing Quantities of given Fluxions, there will be no great Difficulty in conceiving the Reaſons, if what is already delivered in *Sect.* I. *on the direct Method*, has been duly conſidered: Though the Difficulties that occur in this Part, upon another Account, are indeed vaſtly ſuperior.

It is an eaſy Matter, or not impoſſible at moſt, to find the Fluxion of any flowing Quantity whatever; but in the *Inverſe Method* the Caſe is quite different: For, as there is no Method for deducing the Fluent from the Fluxion *a priori*, by a direct Inveſtigation, ſo it is impoſſible to lay down Rules for any other Forms of Fluxions, than thoſe particular ones which we know, from the direct Method, belong to ſuch and ſuch kinds of flowing Quantities. Thus, for Example, the Fluent of $2x\dot{x}$ is known to be x^2, becauſe it is found in *Art.* 6. and 14. that $2x\dot{x}$ is the Fluxion of x^2: But the Fluent of $y\dot{x}$ is unknown, ſince no Expreſſion has been diſcovered that produces $y\dot{x}$ for its Fluxion.

77. Now, as the principal Rule in the *direct Method* is that for the Fluxions of Powers, derived in *Art.* 8. (where it is proved that the Fluxion of x^n is, *univerſally*, expreſſed by $nx^{n-1}\dot{x}$); ſo the moſt general Rule, that can be given in the *Inverſe Method*, muſt be that ariſing from the converſe thereof; which *ſhews how to aſſign the Fluent of any Power of a variable Quantity drawn into the Fluxion of the Root*; and which, expreſſed in Words, will be as follows.

Divide by the Fluxion of the Root, add Unity to the Exponent of the Power, and divide by the Exponent ſo increaſed.

For, dividing the Fluxion $n x^{n-1}\dot{x}$ by $\dot{x}$ (the Fluxion of the Root x) it becomes $n x^{n-1}$; and, adding 1 to the Exponent $(n-1)$ we have $n x^{n}$; which, divided by n, gives x^{n}, the true Fluent of $n x^{n-1}$, by *Art.* 8.

Hence (by the ſame Rule) the

Fluent of $3x^{2}\dot{x}$ will be $= x^{3}$;

That of $8x^{2}\dot{x} = \frac{8x^{3}}{3}$;

That of $2x^{5}\dot{x} = \frac{x^{6}}{3}$

That of $y^{\frac{1}{2}}\dot{y} = \frac{2}{3}y^{\frac{3}{2}}$;

That of $ay^{\frac{5}{3}}\dot{y} = \frac{3ay^{\frac{8}{3}}}{8}$;

That of $y^{\frac{m}{n}}\dot{y} = \frac{y^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{ny^{\frac{m+n}{n}}}{m+n}$;

That of $\frac{a\dot{x}}{x^{n}}$, or $a\dot{x}x^{-n}$, $= \frac{ax^{1-n}}{1-n}$;

That of $\overline{a+z}^{3} \times \dot{z} = \frac{\overline{a+z}^{4}}{4}$;

And that of $\overline{a^{m}+z^{m}}^{n} \times z^{m-1}\dot{z} = \frac{\overline{a^{m}+z^{m}}^{n+1}}{m \times \overline{n+1}}$:

For *here* the Root, or the Quantity under the general Index n, being $a^{m} + z^{m}$, and its Fluxion $= mz^{m-1}\dot{z}$ (*Art.* 14.) we ſhall, by dividing by the laſt of theſe Quantities, have $\frac{\overline{a^{m}+z^{m}}^{n}}{m}$; whence, increaſing the

Index by Unity, and dividing by $(n+1)$ the Index ſo increaſed, there comes out $\dfrac{\overline{a^n+z^m}|^{n+1}}{m \times \overline{n+1}}$.

After the very ſame Manner the Fluents of other Expreſſions may be deduced, when the Quantity, or Multiplicator, without the Vinculum is either equal, or in a conſtant Ratio, to the Fluxion of the Quantity, under the Vinculum: As in the Expreſſion $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$; where the Number of Dimenſions of z under the *Vinculum* (or general Index) being equal to thoſe of z without the *Vinculum* $+ 1$, the Fluent may therefore be had, as in the preceding Examples; and will come out $\dfrac{\overline{a+cz^n}|^{m+1} \times d}{nc \times \overline{m+1}}$: And, that this (or any other Expreſſion derived in like Manner) is the true Fluent will evidently appear, by ſuppoſing x equal to $a+cz^n$ the Quantity under the Vinculum; for then (equal Quantities having equal Fluxions) $\dot{x}$ will be $= ncz^{n-1}\dot{z}$ *; and conſequently $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$ $\left(= x^m \times \dfrac{d\dot{x}}{nc}\right) = \dfrac{dx^m\dot{x}}{nc}$; whoſe Fluent is therefore $\dfrac{dx^{m+1}}{nc \times \overline{m+1}}$ ‡ $= \dfrac{d \times \overline{a+cz^n}|^{m+1}}{nc \times \overline{m+1}}$, *as before.*

* Art. 8.

‡ Art. 77.

78. In aſſigning the Fluents of given Fluxions there is another Particular that ought to be attended to, not yet taken notice of; and that is, whether the flowing Quantity, found by the common Rule, above delivered, does not require the Addition or Subtraction of ſome conſtant Quantity to render it complete. This

indeed can, only, be known from the Nature of the Problem under Conſideration; but that ſuch an Addition or Subtraction may, in ſome Caſes, become neceſſary is evident from the Subject itſelf; ſince a flowing Quantity increaſed, or decreaſed, by a conſtant Quantity, has ſtill the ſame Fluxion; and therefore the Fluent of that Fluxion is as properly expreſſed by the whole compound Expreſſion, as by the variable Part of it, alone: Thus, for Inſtance, the Fluent of $nx^{n-1}\dot{x}$ may be either repreſented by x^n or by $x^n \pm a$, becauſe (a being conſtant) the Fluxion of $x^n \pm a$, as well as of x^n, is $nx^{n-1}\dot{x}$.

79. Hence it appears that it is the variable Part of a Fluent only which is aſſignable by the common Method; the conſtant Part (when ſuch becomes neceſſary) being to be aſcertained from the particular Nature of the Problem. Now to do this, the beſt Way is to conſider how much the variable Part of the Fluent, firſt found, differs from the Truth, in that particular Circumſtance when the required Quantity which the whole Fluent ought to expreſs, is equal to Nothing; then that Difference, added to, or ſubtracted from, the ſaid variable Part, as occaſion requires, will give the Fluent truly corrected: For, ſince the Difference of two Quantities flowing with the ſame Celerity (or having equal Fluxions) is either, Nothing at all, or *conſtantly* the *ſame*, the Difference in that Circumſtance will likewiſe be the Difference in all other Circumſtances: And therefore being added to the leſſer Quantity, or ſubtracted from the greater, both become equal.

80. To render what is above delivered as familiar as may be, I ſhall put down a few Examples; in which the variable Quantities repreſented by x and y are ſuppoſed to begin their Exiſtence together, or to be generated, at the ſame time.

1. Let $\dot{y} = a^2x\dot{x}$; then the Fluent, found as uſual, will be $y = \frac{a^2x^2}{2}$; where taking $y = 0$, $\frac{a^2x^2}{2}$ alſo vaniſhes, (becauſe then $x=0$ by Hypotheſis): Therefore the Fluent requires no Correction in this Caſe.

2. Let $\dot{y} = \overline{a+x}|^3 \times \dot{x}$: Here we firſt have $y = \frac{\overline{a+x}|^4}{4}$; but when $y=0$, then $\frac{\overline{a+x}|^4}{4}$ becomes $= \frac{a^4}{4}$ (ſince x by Hypotheſis is then $= 0$;) Therefore $\frac{\overline{a+x}|^4}{4}$ *always* exceeds y by $\frac{a^4}{4}$; and ſo the Fluent properly corrected will be $y = \frac{\overline{a+x}|^4 - a^4}{4} = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$.

But the very ſame Fluent may be otherwiſe found, without needing any Correction: For the given Equation ($\dot{y} = \overline{a+x}|^3 \times \dot{x}$), by expanding $\overline{a+x}|^3$, is tranſformed to $\dot{y} = a^3\dot{x} + 3a^2x\dot{x} + 3ax^2\dot{x} + x^3\dot{x}$; whence $y = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$; the ſame *as above.*

Hence it appears that the Fluent of an Expreſſion, found according to one Form, may require a very different Correction from the Fluent of the ſame Fluxion found according to another Form.

3. Let $\dot{y} = \overline{a^2-x^2}|^{\frac{1}{2}} \times x\dot{x}$; then, firſt, $y = -\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$; where taking $y=0$, $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$ becomes

$=$

$= -\frac{a^3}{3}$; therefore $-\frac{\overline{a^2-x^2}\rvert^{\frac{3}{2}}}{3}$ is too little by $\frac{a^3}{3}$; and ſo the Fluent corrected will be $y = \frac{a^3}{3} - \frac{\overline{a^2-x^2}\rvert^{\frac{3}{2}}}{3}$.

4. Let $\dot{y} = \overline{a^m+x^m}\rvert^{n} \times x^{m-1}\dot{x}$: Here we firſt have $y = \frac{\overline{a^m+x^m}\rvert^{n+1}}{m\times\overline{n+1}}$; and making $y=0$, the latter Part of the Equation becomes $\frac{\overline{a^m}\rvert^{n+1}}{m\times\overline{n+1}} = \frac{a^{mn+m}}{m\times\overline{n+1}}$; whence the Equation, or Fluent, truly corrected is $y = \frac{\overline{a^m+x^m}\rvert^{n+1} - a^{mn+m}}{m\times\overline{n+1}}$.

5. Laſtly, let $\dot{y} = \overline{a+bx^m+cx^n}\rvert^{p} \times \overline{mbx^{m-1}\dot{x}+ncx^{n-1}\dot{x}}$; then, in the firſt Place, we have $y = \frac{\overline{a+bx^m+cx^n}\rvert^{p+1}}{p+1}$; which corrected, as above, becomes $y = \frac{\overline{a+bx^m+cx^n}\rvert^{p+1} - a^{p+1}}{p+1}$.

81. Hitherto x and y are both ſuppoſed equal to Nothing at the ſame time; but that will not always be the Caſe in the Solution of Problems. Thus, for Inſtance, though the Sine and Tangent of an Arch are both equal to Nothing when the Arch itſelf is equal to Nothing, yet

the Secant is then equal to the Radius: It will be proper therefore to add an Example or two wherein the Value of y is equal to Nothing, when that of x is equal to any given Quantity a.

Let, then, the Equation $\dot{y} = x^2\dot{x}$ be firſt propoſed; whereof the Fluent (firſt taken) is $y = \frac{x^3}{3}$; but when $y = 0$, then $\frac{x^3}{3} = \frac{a^3}{3}$, by Hypotheſis; therefore the Fluent, corrected, is $y = \frac{x^3 - a^3}{3}$.

Again, let the propoſed Equation be $\dot{y} = -x^n\dot{x}$; then will $y = -\frac{x^{n+1}}{n+1}$; which corrected becomes $y = \frac{a^{n+1} - x^{n+1}}{n+1}$.

Laſtly, let $\dot{y} = \overline{c^3 + bx^2}\big|^{\frac{1}{2}} \times x\dot{x}$; then, firſt, $y = \frac{\overline{c^3+bx^2}\big|^{\frac{3}{2}}}{3b}$; and, when $y = 0$ and $x = a$, $\frac{\overline{c^3+bx^2}\big|^{\frac{3}{2}}}{3b}$ becomes $= \frac{\overline{c^3+ba^2}\big|^{\frac{3}{2}}}{3b}$: therefore the Fluent corrected is $y = \frac{\overline{c^3+bx^2}\big|^{\frac{3}{2}} - \overline{c^3+ba^2}\big|^{\frac{3}{2}}}{3b}$.

82. All the Examples hitherto given relate to ſuch Fluxions as involve one variable Quantity only in each Term, whoſe Fluents are aſſignable from the Converſe of the firſt General Rule, *in Section* I. But, beſides theſe, various other Forms of Fluxions may be propoſed, involving two or more variable Quantities, whoſe Fluents may alſo be found by Help of the other two General Rules delivered in the ſame Section.

Thus the Fluent of $y\dot{x}+x\dot{y}$ is expreſſed by xy*; that of $\frac{y\dot{x}-x\dot{y}}{y^2}$ by $\frac{x}{y}$†; that of $a\dot{x}+x\dot{y}+y\dot{x}$ by $ax+xy$‡; and that of $\overline{nx\dot{y}y^{n-1}+y^n\dot{x}-nax^{n-1}\dot{x}}\times\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$ by $\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$: For, dividing (in the laſt Caſe) by the Fluxion of the Root $\overline{y^nx-ax^n}$*, which (by *Art.* 14 and 15) is $nxy^{n-1}\dot{y}+y^n\dot{x}-nax^{n-1}\dot{x}$, we firſt have $\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$; whence, adding Unity to the Exponent $\frac{p}{m}$, and dividing by the Exponent ſo increaſed, we get

$$\frac{\overline{y^nx-a^{x^n}}\big|^{\frac{p}{m}+1}}{\frac{p}{m}+1}=\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$$

for the true Fluent of the Quantity propoſed. But it ſeldom happens that theſe Kinds of Fluxions which involve two different variable Quantities in one Term, and yet admit of known, or perfect, Fluents, are to be met with in Practice: I ſhall therefore take no further Notice of them in this Place (but refer the Reader to the ſecond Part of the Work) my Deſign here being to inſiſt only upon what is moſt general and uſeful in the Subject; which brings me to further conſider thoſe Forms of Fluxions, involving one variable Quantity only, that frequently occur in the Solution of Problems, whoſe Fluents may (after proper Transformation) be found, by the Rule already delivered in *Art.* 77.

* Art. 10.

† Art. 13.

‡ Art. 10.

* Art. 77.

83. It has been already hinted, that if a Fluxion of the Binomial Kind, as $\overline{a+cz^{n}}^{\,m} \times dz^{n-1}\dot{z}$, has the Index $(n-1)$ of the variable Quantity (z) without the *Vinculum* $+ 1$, equal to (n) the Index of the ſame Quantity under the *Vinculum*, the Fluent thereof may be then truly found by the forementioned Rule. But the ſame Obſervation may be farther extended to *thoſe Caſes where the Index without the Vinculum increaſed by Unity is equal to any Multiple of that under the Vinculum*; as in the Expreſſions, $\overline{a+cz^{n}}^{\,m} \times dz^{2n-1}\dot{z}$, $\overline{a+cz^{n}}^{\,m} \times dz^{3n-1}\dot{z}$, $\overline{a+cz^{n}}^{\,m} \times dz^{4n-1}\dot{z}$, &c. Whoſe Fluents are thus determined.

Put $a+cz^{n} = x$, then will $z^{n} = \frac{x-a}{c}$, and $nz^{n-1}\dot{z} = \frac{\dot{x}}{c}$*; and therefore $z^{2n-1}\dot{z} = \frac{x-a}{c} \times \frac{\dot{x}}{nc} = \frac{x\dot{x}-a\dot{x}}{ncc}$; whence by Subſtitution we get $\overline{a+cz^{n}}^{\,m} \times dz^{2n-1}\dot{z} = \frac{x^{m} \times d \times \overline{x\dot{x}-a\dot{x}}}{nc^{2}} = d \times \frac{x^{m+1}\dot{x}-ax^{m}\dot{x}}{nc^{2}}$; Whoſe Fluent (by *Art.* 77.) is therefore $= \frac{d}{nc^{2}} \times \overline{\frac{x^{m+2}}{m+2} - \frac{ax^{m+1}}{m+1}}$; which, by reſtoring the Value of x, becomes $\frac{d}{nc^{2}} \times \overline{\frac{\overline{a+cz^{n}}^{\,m+2}}{m+2} - \frac{a \times \overline{a+cz^{n}}^{\,m+1}}{m+1}} =$

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* Art. 8.

$\frac{d\times\overline{a+cz^n}^{m+1}}{nc^2} \times \overline{\frac{a+cz^n}{m+2} - \frac{a}{m+1}} = \frac{d\times\overline{a+cz^n}^{m+1}}{nc^2} \times \overline{\frac{cz^n}{m+2} - \frac{a}{\overline{m+2}\times\overline{m+1}}}$; the true Fluent of $\overline{a+cz^n}^{m} \times dz^{2n-1}\dot{z}$.

Again; for the Fluent of $\overline{a+cz^n}^{m} \times dz^{3n-1}\dot{z}$, becauſe $z^{n-1}\dot{z} = \frac{\dot{x}}{nc}$, and $z^n = \frac{x-a}{c}$, we have $z^{3n-1}\dot{z}$ $\left(= z^{2n} \times z^{n-1}\dot{z}\right) = \frac{\overline{x-a}^2}{c^2} \times \frac{\dot{x}}{nc} = \frac{x^2\dot{x} - 2ax\dot{x} + a^2\dot{x}}{nc^3}$: Whence, $\overline{a+cz^n}^{m}$ being $= x^m$, we get $\overline{a+cz^n}^{m} \times dz^{3n-1}\dot{z} = dx^m \times \frac{x^2\dot{x} - 2ax\dot{x} + a^2\dot{x}}{nc^3} = \frac{d}{nc^3} \times \overline{x^{m+2}\dot{x} - 2ax^{m+1}\dot{x} + a^2x^m\dot{x}}$; whoſe Fluent is therefore $= \frac{d}{nc^3} \times \overline{\frac{x^{m+3}}{m+3} - \frac{2ax^{m+2}}{m+2} + \frac{a^2x^{m+1}}{m+1}} = \frac{dx^{m+1}}{nc^3} \times \overline{\frac{x^2}{m+3} - \frac{2ax}{m+2} + \frac{a^2}{m+1}} = \frac{d\times\overline{a+cz^n}^{m+1}}{nc^3} \times \overline{\frac{\overline{a+cz^n}^2}{m+3} - \frac{2aa+2acz^n}{m+2} + \frac{a^2}{m+1}} = \frac{d\times\overline{a+cz^n}^{m+1}}{nc^3} \times \overline{\frac{c^2z^{2n}}{m+3} - \frac{2acz^n}{\overline{m+3}\times\overline{m+2}} + \frac{2a^2}{\overline{m+3}\times\overline{m+2}\times\overline{m+1}}}$.

Univerſally, let r denote any whole poſitive Number whatever, and let the Fluent of $\overline{a+cz^{n}}|^{m} \times dz^{rn-1}\dot{z}$ be required; then, by putting $a+cz^{n}=x$, and proceeding as above, our propoſed Fluxion is transformed to $\frac{dx^{m}\dot{x}}{nc^{r}} \times \overline{x-a}|^{r-1}$; which, expanding $\overline{x-a}|^{r-1}$ (by the Binomial Theorem) becomes

$$\frac{d}{nc^{r}} \times \overline{x^{m+r-1}\dot{x} - \overline{r-1} \times ax^{m+r-2}\dot{x} + \overline{r-1} \times \frac{r-2}{2} \times a^{2}x^{m+r-3}\dot{x}} \; \frac{\quad}{m+r}$$

&c. whoſe Fluent is therefore

$$= \frac{d}{nc^{r}} \times \overline{\frac{x}{m+r} - \frac{\overline{r-1} \times ax^{m+r-1}}{m+r-1} + \frac{\overline{r-1} \times \overline{r-2} \times a^{2}x^{m+r-2}}{2 \times \overline{m+r-2}}} \; \&c. =$$

$$\frac{dx^{m+1}}{nc^{r}} \times \overline{\frac{x^{r-1}}{m+r} - \frac{\overline{r-1} \times ax^{r-2}}{m+r-1} + \frac{\overline{r-1} \times \overline{r-2} \times a^{2}x^{r-3}}{2 \times \overline{m+r-2}}}$$

&c.

Where, r being a whole poſitive Number, the Multiplicators $1, \overline{r-1}, \overline{r-1} \times \overline{r-2}, \overline{r-1} \times \overline{r-2} \times \overline{r-3}$, &c. will therefore become equal to Nothing, after the r firſt terms; and ſo, the Series terminating, the Fluent itſelf will be truly exhibited in that Number of Terms: Except when $m+r$ is likewiſe a whole poſitive Number, leſs than r; in which Circumſtance the Diviſors $m+r$, $m+r-1$, $m+r-2$, &c. becoming equal to Nothing, before the Multiplicators, the correſponding Terms of the Series will be infinite. And in that Caſe the Fluent is ſaid to fail, ſince Nothing can then be determined from it.

84. Beſides the foregoing, there is another Way of deriving the Fluent of $\overline{a+cz^{n}}^{m} \times dz^{rn-1}\dot{z}$, in Terms of the original flowing Quantity z; which will afford a Theorem more commodious for Practice than that above given: The Method of Inveſtigation is thus.

Let $d \times \overline{a+cz^{n}}\rceil^{m+1} \times Az^{p} + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}$ &c. (where p, v, A, B, C, &c. denote unknown, but determinate, Quantities) be aſſumed for the Fluent ſought: Then by taking the Fluxion of the Quantity ſo aſſumed we ſhall have

$dcn \times \overline{m+1} \times z^{n-1}\dot{z} \times \overline{a+cz^{n}}\rceil^{m} \times \overline{Az^{p} + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}\ \&c.} + d \times \overline{a+cz^{n}}\rceil^{m+1} \times \overline{pAz^{p-1}\dot{z} + \overline{p-v} \times Bz^{p-v-1}\dot{z} + \overline{p-2v} \times Cz^{p-v-1}\dot{z}\ \&c.}$ * which being put equal to the given Fluxion, $\overline{a+cz^{n}}\rceil^{m} \times dz^{rn-1}\dot{z}$, and the whole Equation divided by $\overline{a+cz^{n}}\rceil^{m} \times dz^{-1}\dot{z}$, there comes out

$$\left.\begin{array}{l} +cn \times \overline{m+1} \times z^{n} \times \overline{Az^{p} + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}\ \&c.} \\ +\overline{a+cz^{n}} \times \overline{pAz^{p} + \overline{p-v} \times Bz^{p-v} + \overline{p-2v} \times Cz^{p-2v}\ \&c.} \end{array}\right\} = z^{rn}$$

Whence, by collecting the Coefficients of the like Powers of z, we have

$$\left.\begin{array}{llll} \left.\begin{array}{r} n \times \overline{m+1} \\ +p \end{array}\right\} \times cAz^{p+n} & + \left.\begin{array}{r} n \times \overline{m+1} \\ +p-v \end{array}\right\} \times cBz^{p+n-v} & + \left.\begin{array}{r} n \times \overline{m+1} \\ +p-2v \end{array}\right\} \times cCz^{p+n-2v} & \&c. \\ -z^{rn} & + \quad paAz^{p} & + \overline{p-v} \times aBz^{p-v} & \&c. \end{array}\right\} = 0$$

Where, comparing $p+n$ and rn, the two greateſt Exponents of z, we find $p = rn - n = \overline{r-1} \times n$; and by comparing the two next inferior Exponents $p+n-v$, and p, we likewiſe

*Art. 8, 10.

likewiſe get $v=n$; which Values being ſubſtituted above, our Equation is reduced to

$$\left.\begin{aligned}&\overline{m+r}\times ncAz^{rn}+\overline{m+r-1}\times ncBz^{rn-n}+\overline{m+r-2}\times ncCz^{rn-2n}\ \&c.\\ &-z^{rn}+\overline{r-1}\times naAz^{rn-n}+\overline{r-2}\times naBz^{rn-2n}\ \&c.\end{aligned}\right\}=0$$

Where, putting $m+r=s$, and comparing the Coefficients of the homologous Terms *, we have $A=\frac{1}{snc}$, $B=-\frac{\overline{r-1}\times aA}{\overline{s-1}\times c}=-\frac{\overline{r-1}\times a}{s\times\overline{s-1}\times nc^2}$, $C=-\frac{\overline{r-2}\times aB}{\overline{s-2}\times c}=\frac{\overline{r-1}\times\overline{r-2}\times a^2}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}$, $D=-\frac{\overline{r-3}\times aC}{\overline{s-3}\times c}$ $=-\frac{\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times a^3}{s\times\overline{s-1}\times\overline{s-2}\times\overline{s-3}\times nc^4}$, &c. &c. which Values, with thoſe of p and v, being ſubſtituted in the aſſumed Fluent, it becomes $d\times\overline{a+cz^n}\rceil^{m+1}\times\overline{\frac{z^{rn-n}}{snc}-\frac{\overline{r-1}\times az^{rn-2n}}{s\times\overline{s-1}\times nc^2}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}\ \&c.}$ $=\frac{d\times\overline{a+cz^n}\rceil^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}\ \&c.}$ the true Fluent of $\overline{a+cz^n}\rceil^{m+1}\times dz^{rn-1}\dot{z}$, which was to be determined: Which Fluent therefore, when r is a whole poſitive Number, will always terminate in as many Terms as are expreſſed by that Number; except in that particular Caſe, ſpecified in the laſt Article. Thus, if $r=2$, or the

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Vid. p. 181. *of my Treatiſe of Algebra.*

the given Fluxion be $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$; then, s $(m+r)$ being $=m+2$, the Fluent itſelf will become

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+2}} \times \overline{\frac{z^n}{1} - \frac{a}{\overline{m+1}\times c}} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times \overline{\frac{cz^n}{m+2} - \frac{a}{\overline{m+2}\times\overline{m+1}}}$$

; which is exactly the ſame with the firſt of thoſe found in *Art.* 83. by a different Method.

The like Agreement will likewiſe be found, when r is $=3$: But when r, either denotes a broken, or a negative, Number; the Series for the Fluent will then run on to Infinity; becauſe no one of the Multiplicators $r-1$, $r-2$, $r-3$, $r-4$, *&c.* can in that Caſe be equal to Nothing.

85. The foregoing Fluent, it may be obſerved, was found by aſſuming $d\times\overline{a+cz^n}|^{m+1} \times Az^p + Bz^{p-v} + Cz^{p-2v}$ *&c.* and comparing the two greateſt Exponents, of the Equation thence reſulting: But if, inſtead of $Az^p + Bz^{p-v} + Cz^{p-2v}$ *&c.* an aſcending Series, as $Az^p + Bz^{p+v} + Cz^{p+2v}$ *&c.* (where the Exponents of z continually increaſe) be taken, and the two leaſt Indices of z in the Equation (in like Manner reſulting) be compared together, the ſame Fluent will be had according to a different Form, which will be of good Uſe in many Caſes, when the foregoing fails, or runs out into an Infinite Series.

Thus, if $p+v$, $p+2v$, *&c.* be wrote in the Room of $p-v$, $p-2v$, *&c.* reſpectively, in the firſt Equation of the laſt Article, it will appear that

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^{n}\times\overline{Az^{p}+Bz^{p+u}+Cz^{p+2v}\ \text{\&c.}} \\ +\overline{a+cz^{n}}\times pAz^{p}+\overline{p+v}\times Bz^{p+v}+\overline{p+2v}\times Cz^{p+2v}\ \text{\&c.} \end{array}\right\} = z^{rn}$$

Which Equation may be reduced to

$$\left.\begin{array}{l} paAz^{p}+\overline{p+v}\times aBz^{p+v}+\overline{p+2v}\times aCz^{p+2v}\ \text{\&c.} \\ -z^{rn}+\left.\begin{array}{l} n\times\overline{m+1} \\ +p \end{array}\right\}\times cAz^{p+n}+\left.\begin{array}{l} n\times\overline{m+1} \\ +p+v \end{array}\right\}\times cBz^{p+n+v}\ \text{\&c.} \end{array}\right\} = 0$$

Where, by comparing the two leaſt Exponents, &c. p will be found $= rn$, $v = n$; $A = \frac{1}{pa} = \frac{1}{rna}$; $B = -\frac{\overline{p+n\times\overline{m+1}}\times cA}{\overline{p+v}\times a} = -\frac{\overline{r+m+1}\times ncA}{\overline{r+1}\times na} = -\frac{\overline{r+m+1}\times c}{r\times\overline{r+1}\times na^{2}}$; $C = -\frac{\overline{p+v+n\times\overline{m+1}}\times cB}{\overline{p+2v}\times a} = -\frac{\overline{r+m+2}\times ncB}{\overline{r+2}\times na} = \frac{\overline{r+m+1}\times\overline{r+m+2}\times c^{2}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}$ &c. &c.

Therefore, denoting $r+m$ by s (as above) the Fluent of $\overline{a+cz^{n}}^{m}\times dz^{rn-1}\dot{z}$, will (*alſo*) be truly repreſented by

$$d\times\overline{a+cz^{n}}^{m+1}\times\overline{\frac{z^{rn}}{rna}-\frac{\overline{s+1}\times cz^{rn+n}}{r\times\overline{r+1}\times na^{2}}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{rn+2n}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}\ \text{\&c.}}$$ or its Equal $$\frac{\overline{a+cz^{n}}^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^{n}}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^{2}}\ \text{\&c.}}$$

Which Series will terminate when s (or $r+m$) is a whole negative Number; and therefore in all ſuch Caſes the Fluent is exactly determined; provided r be not alſo a negative Integer leſs than s; for in this particular Circumſtance the Fluent fails, the Diviſor firſt becoming equal to Nothing. *Vid. Art.* 83.

The Uſe of the two foregoing general Expreſſions, for the Fluent of $\overline{a+cz^n}|^{m}\times dz^{rn-1}\dot{z}$, will appear from the following Examples.

EXAMPLE I.

86. *Let it be requir'd to find the Fluent of* $\frac{bx\dot{x}}{\overline{a+x}|^{\frac{1}{2}}}$, or

$$\overline{a+x}|^{-\frac{1}{2}}\times bx\dot{x}.$$

By comparing the Fluxion here propoſed with $\overline{a+cz^n}|^{m}\times dz^{rn-1}\dot{z}$, we have $a=a$, $c=1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $d=b$, $rn-1$ (or $r-1$) $=1$; whence $r=2$, and s $(r+m)=\frac{3}{2}$; whereof the former being a whole poſitive Number, let theſe Values be therefore ſubſtituted in

$$\left(\frac{d\times\overline{a+cz^n}|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}},\ \&c.\right)$$

the firſt of the two general Expreſſions for the Fluent, and it will become $\frac{b\times\overline{a+x}|^{\frac{3}{2}}}{\frac{3}{2}}\times\overline{x-\frac{a}{\frac{1}{2}}}=\frac{b\times\overline{a+x}|^{\frac{3}{2}}\times\overline{2x-4a}}{3}$, the Quantity ſought in this Caſe.

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EXAMPLE II.

87. *Let the Fluxion propoſed be* $\frac{b\dot{x}x^{3n-1}}{\sqrt{a+fx^n}}$, or

$$\overline{a+fx^n}\Big|^{-\frac{1}{2}}\times bx^{3n-1}\dot{x}.$$

Here, by proceeding as above, we have $a=a$, $c=f$, $z=x$, $n=n$, $m=-\frac{1}{2}$, $d=b$, $r=3$, and s $(r+m)=\frac{5}{2}$: Whence, by ſubſtituting theſe ſeveral Values in the ſame general Expreſſion, we get $\frac{b\times\overline{a+fx^n}\big|^{\frac{1}{2}}}{\frac{5}{2}nf}\times$

$$\overline{x^{2n}-\frac{2ax^n}{\frac{3}{2}f}+\frac{2a^2}{\frac{3}{2}\times\frac{1}{2}f^2}}=\frac{b\times\overline{a+fx^n}\big|^{\frac{1}{2}}}{nf^3}\times\frac{\overline{6f^2x^{2n}-8afx^n+16a^2}}{15}.$$

EXAMPLE III.

88. *Wherein the Quantity propoſed is* $\frac{\dot{y}\sqrt{g^2+y^2}}{y^6}$, or

$$\overline{g^2+y^2}\Big|^{\frac{1}{2}}+y^{-6}\dot{y}.$$

Here we have $a=g^2$, $c=1$, $z=y$, $n=2$, $m=\frac{1}{2}$, $d=1$, $rn-1$ (or $2r-1$) $=-6$; whence r $\left(=\frac{-6+1}{2}\right)$ $=-\frac{5}{2}$, and s $(r+m)=-2$; whereof the latter being a whole Negative Number, let the ſeveral Values here exhibited be therefore ſubſtituted in

$a+$

$$\left(\frac{\overline{a+cz^{n}}\rceil^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^{n}}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^{2}}}\right.$$

&c.) the latter of the two general Expreſſions above derived, and it will become $\frac{\overline{g^{2}+y^{2}}\rceil^{\frac{3}{2}}\times y^{-5}}{-5g^{2}}\times$ $\overline{1-\frac{-1\times y^{2}}{-\frac{3}{2}\times g^{2}}}=\frac{\overline{g^{2}+y^{2}}\rceil^{\frac{3}{2}}\times\overline{2y^{2}-3g^{2}}}{15g^{4}y^{5}}$; the true Fluent required.

EXAMPLE IV.

89. *Laſtly, let the given Fluxion be* $\overline{a-fz^{n}}\rceil^{\frac{1}{2}}\times$ $z^{-\frac{7}{2}n-1}\dot{z}$.

Then, a being $=a$, $c=-f$, $m=\frac{1}{2}$, $d=1$, $r=-\frac{7}{2}$, and the reſt as in the general Fluxion $\overline{a+cz^{n}}\rceil^{m}\times$ $dz^{rn-1}\dot{z}$; we ſhall, by ſubſtituting in the ſecond Form (becauſe s is here equal to (-3) a whole negative Number) have $\frac{\overline{a-fz^{n}}\rceil^{\frac{3}{2}}\times z^{-\frac{7}{2}n}}{-\frac{7}{2}na}\times\overline{1-\frac{-2\times-fz^{n}}{-\frac{5}{2}a}}$ $\overline{\frac{-2\times-1\times f^{2}z^{2n}}{-\frac{5}{2}\times-\frac{3}{2}a^{2}}}=\frac{\overline{a-fz^{n}}\rceil^{\frac{3}{2}}}{-\frac{7}{2}naz^{\frac{7}{2}n}}\times\overline{1+\frac{4fz^{n}}{5a}+\frac{8f^{2}z^{2n}}{15a^{2}}}$

$$=-\frac{\overline{a-fz^{n}}\rceil^{\frac{3}{2}}\times\overline{30a^{2}+24afz^{n}+16f^{2}z^{2n}}}{105na^{3}z^{\frac{7}{2}n}}.$$

90. Having inſiſted largely on the Manner of finding ſuch Fluents as can be truly exhibited in Algebraic Terms; it remains now to ſay ſomething with regard

to thoſe other Forms of Expreſſions, involving one variable Quantity only, which, *yet*, are ſo affected by compound Diviſors and radical Quantities, that their Fluents cannot be *accurately* determined by any Method whatſoever; of which there are innumerable Kinds; But there is one general Method whereby the Fluents of ſuch Expreſſions are approximated, to any aſſigned Degree of Exactneſs; namely, the Method of *Infinite Series*; which it will, therefore, be neceſſary to explain; ſo far as relates to the Manner of expounding the Value of any compound Fraction, or ſurd Quantity, by Help of ſuch a Series.

EXAMPLE I.

91. *Let, then, the Fraction* $\frac{ax}{a-x}$ *be, firſt, given; to be converted into an Infinite Series.*

Divide the Numerator ax by the Denominator $a-x$, as is taught in Compound Diviſion of common Algebra; then the Operation will ſtand as follows;

$$\begin{array}{l}
a-x)\,ax\,\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\&c.\right. \\
\quad \underline{ax-xx} \\
\qquad +xx \\
\qquad \underline{+xx-\frac{x^3}{a}} \\
\qquad\qquad +\frac{x^3}{a} \\
\qquad\qquad \underline{+\frac{x^3}{a}-\frac{x^4}{a^2}} \\
\qquad\qquad\qquad +\frac{x^4}{a^2} \\
\qquad\qquad\qquad \&c.
\end{array}$$

Where the Quotient, or Series $x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} + \frac{x^6}{a^5}$ *&c.* infinitely continued, is taken to expound the Value of the propoſed Fraction $\frac{ax}{a-x}$.

92. But, though the Series thus ariſing ought to be carried on to an Infinity of Terms, to have the true Value of the Quantity firſt propoſed; or, though the Quotient, continued to ever ſo great a Number of Terms, will be *ſtill* ſomething defective of the Truth; yet, if the Value of the Quantity (x) in the Numerator be but ſmall in Compariſon of the Quantity (a) in the Denominator, the Remainder, after a few Terms in the Quotient, will become ſo exceeding ſmall, as to be neglected without any conſiderable Error; and then the Value of the *Whole*, or of the Quantity firſt propoſed, will be, very nearly, exhibited, by taking a ſmall Number of the leading Terms only.

Thus, for Inſtance, let the Value of a be expounded by 10, and that of x by Unity; then the Remainder $\left(\frac{x^3}{a}\right)$ after the two firſt Terms of the Quotient, being $= \frac{1}{10}$, this Value, divided by the given Diviſor ($a-x=$) 9, will therefore give $\frac{1}{90} = 0{,}01111111$, *&c.* for the Defect, by taking the two firſt Terms only: But, if the three firſt Terms be taken, the Defect will be *ſtill* leſs conſiderable; amounting to no more than $\frac{1}{900}$, or 0,00111111, *&c.*

This may likewiſe be made to appear, without any regard to the Remainder, by collecting into one Sum, the Values of all the Terms to be taken: For, if only the firſt two ($x + \frac{x^2}{a}$) be propoſed, their Sum will be

= 1, 1; which, deducted from the true Value of the given Fraction $\frac{ax}{a-x}\left(=\frac{10}{9}\right) = 1,1111111$ &c. the Difference will come out 0.01111, *the very ſame as before.*

Thus, alſo, by collecting the Sum of the three, four and five, &c. firſt Terms of the Series, you will have 1,11; 1,111; and 1,1111 &c. which, being ſucceſſively deducted from 1,11111111 &c. (as above) there will remain 0,001111 &c. 0,0001111 &c. 0,00001111 &c. for the Errors or Defects in thoſe Caſes reſpectively.

93. From what has been ſaid in the preceding Article it appears, that Infinite Serieſes, in Algebra (according to a common Obſervation) are ſimilar to, or correſpond with, Decimal Fractions in common Arithmetick: For, as a Decimal Fraction may be carry'd on to any propoſed Number of Places, however great, and yet never amount to a Quantity, which but a very little exceeds the Value of the three or four firſt Places; ſo a Series may be infinite with regard to the Number of its Terms, and yet a few of the leading Terms only, may be ſufficient to expreſs the Value of the *Whole*, very nearly: Provided, always, that the Series has a ſufficient Rate of Convergency, or that its Terms decreaſe in a pretty large Proportion: For, otherwiſe, *even*, a great Number of Terms may be uſed to little Purpoſe: Thus, in the foregoing Series, $x + \frac{x^2}{a} + \frac{x^3}{a^2}$ &c. if x be taken $= a$, no Number of Terms will be ſufficient to exhibit the Value of the correſponding Fraction $\frac{ax}{a-x}$, it being infinite in that Circumſtance.

94. Having endeavoured to ſhew, that the true Value of an infinite Series may be nearly obtained by adding together a few of the firſt Terms only, I ſhall now proceed to give other Examples of the Manner of

converting fractional, and ſurd, Quantities into ſuch Kinds of Serieſes, in order to the Approximation of the Fluents of Expreſſions affected by them.

EXAMPLE II.

Let the Quantity propoſed be the Fraction $\frac{c^2}{c^2+2cy+y^2}$; then, by proceeding as in the firſt Example, you will have

$$\begin{array}{l} c^2+2cy+y^2)\ c^2 \ldots\ldots \left(1-\frac{2y}{c}+\frac{3y^2}{c^2}-\frac{4y^3}{c^3}\ \&c. \right. \\ \qquad\qquad \underline{c^2+2cy+y^2} \\ \qquad\qquad\quad -2cy-y^2 \\ \qquad\qquad\quad \underline{-2cy-4y^2-\frac{2y^3}{c}} \\ \qquad\qquad\qquad\quad +3y^2+\frac{2y^3}{c}\ \&c. \end{array}$$

Where, from a few of the firſt Terms of the Quotient, the Law of Continuation is manifeſt; the Numerators being in Arithmetical Progreſſion; and the Signs, + and —, alternately.

EXAMPLE III.

95. *Let the Quantity given be* $\frac{1+x^2-2x^4}{1-x-x^2}$.

Then the Quotient will be $1+x+3x^2+4x^3+5x^4+9x^5+14x^6$ &c. where the *Law of Continuation* is manifeſt; being ſuch that the Coefficient of each ſucceeding Term is equal to the Sum of thoſe of the two Terms immediately preceding it.

EXAMPLE IV.

96. *Let the Radical Quantity* $\sqrt{a^2+x^2}$ *be propoſed.*

Here, according to the common Method of extracting the Square Root, the Proceſs will ſtand as follows:

$$
\begin{array}{r}
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\, aa+xx\, \Big(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}\ \text{\&c.} \\
aa \\
\hline
+xx \\
+xx+\frac{x^4}{4a^2} \\
\hline
-\frac{x^4}{4a^2} \\
-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\ \text{\&c. \&c.}
\end{array}
$$

97. The Law of Continuation in Serieſes, thus ariſing, from radical Quantities, is not eaſily diſcovered: But, if you would carry on the Series to any propoſed Number of Terms, the Work will be a good deal ſhortned, by dividing the Remainder by the Diviſor, when half that Number of Terms is found (as in common Diviſion) and obſerving, at the ſame time, to neglect all ſuch Terms whoſe Indices would exceed the greateſt, or the greateſt Plus the common Difference, in the ſaid Remainder, according as the whole Number of Terms propoſed to be found is odd, or even.

Thus, if it were propoſed to continue the foregoing Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$ to 6 Terms, then the Diviſor (or double Quotient) being $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, and the Remainder $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ (as appears from the laſt Article) the reſt of the Operation will ſtand thus:

$$
\begin{array}{l}
2a + \frac{x^2}{a} - \frac{x^4}{4a^3}\Big)\ \frac{x^6}{8a^4} - \frac{x^8}{64a^6} + 0\ \Big(\frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} \\
\qquad\qquad\qquad \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} \\
\hline
\qquad\qquad\qquad\quad - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} \\
\qquad\qquad\qquad\quad - \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8} \\
\hline
\qquad\qquad\qquad\qquad\qquad + \frac{7x^{10}}{128a^8}
\end{array}
$$

Which three Terms thus found being added to thoſe found above, we have $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, for the 6 firſt Terms of an infinite Series exhibiting the Value of $\sqrt{a^2 + x^2}$.

98. Another Way of reſolving any radical Quantity, is to aſſume a Series (with unknown Coefficients) for the Value thereof; and then the Series ſo aſſumed being raiſed to the ſecond, third, or fourth Power, *&c.* according as the Root to be extracted is a ſquare, cubic, or biquadratic one, *&c.* an Equation will be obtained (free from Surds) from whence, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Series ſought, will be determined; as in

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EXAMPLE V.

Where it is propoſed to extract the Square Root of $a^{2n}+x^{2n}$ *in an Infinite Series.*

In which Caſe, aſſuming $A+Bx^{2n}+Cx^{4n}+Dx^{6n}+$ $+Ex^{8n}$ &c. for the required Series, and taking the Square thereof, we have

$$\left.\begin{array}{llll} A^2+2ABx^{2n}+2ACx^{4n} & +2ADx^{6n} & +2AEx^{8n}\,\&c. \\ \qquad\qquad\quad +B^2x^{4n} & +2BCx^{6n} & +2BDx^{8n}\,\&c. \\ & & +C^2x^{8n}\,\&c. \end{array}\right\} = a^{2n}+x^{2n};$$

and conſequently

$$\left.\begin{array}{llllll} A^2 & +2ABx^{2n} & +2ACx^{4n} & +2ADx^{6n} & +2AEx^{8n}\,\&c. \\ -a^{2n} & -x^{2n} & +B^2x^{4n} & +2BCx^{6n} & +2BDx^{8n}\,\&c. \\ & & & & +C^2x^{8n}\,\&c. \end{array}\right\} = 0.$$

Therefore $A^2-a^{2n}=0$, $2AB-1=0$, $2AC+B^2=0$, $2AD+2BC=0$, $2AE+2BD+C^2=0$, * &c. From which we get $A=a^n$; $B\left(=\frac{1}{2A}\right)=\frac{1}{2a^n}$; $C\left(=-\frac{B^2}{2A}\right)=-\frac{1}{8a^{3n}}$; $D\left(=-\frac{BC}{A}\right)=\frac{1}{16a^{5n}}$; $E\left(=-\frac{2BD+C^2}{2A}\right)=-\frac{5}{128a^{7n}}$ &c. whence we have $A+Bx^{2n}+Cx^{4n}+Dx^{6n}$ &c. $\left(=\sqrt{a^{2n}+x^{2n}}\right)=a^n$

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* *Vid. p.* 181 *of my Treatiſe of Algebra.*

$+ \frac{x^{2n}}{2a^{n}} - \frac{x^{4n}}{8a^{3n}} + \frac{x^{6n}}{16a^{5n}} - \frac{5x^{8n}}{128a^{7n}}$ *&c.* Which Series, if *n* be expounded by Unity, will become $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$ *&c.* the very ſame with that in the preceding Article found by the common Method.

EXAMPLE VI.

99. *Let it be required to reſolve* $\overline{a+bx^{n}}\big|^{\frac{1}{3}}$ *into an Infinite Series.*

Here, by aſſuming $A + Bx^{n} + Cx^{2n} + Dx^{3n}$ *&c.* and cubing the ſame, *&c.* we have

$$\left.\begin{array}{l} A^3 + 3A^2Bx^{n} + 3A^2Cx^{2n} + 3A^2Dx^{3n} + \&c. \\ -a - bx^{n} \quad\;\; + 3AB^2x^{2n} + 6ABCx^{3n} + \&c. \\ \qquad\qquad\qquad\qquad + \quad B^3x^{3n} + \&c. \end{array}\right\} = 0$$

Therefore $A = a^{\frac{1}{3}}$; $B \left(= \frac{b}{3A^2}\right) = \frac{b}{3a^{\frac{2}{3}}}$; $C \left(= -\frac{B^2}{A}\right) = -\frac{b^2}{9a^{\frac{5}{3}}}$; $D \left(= -\frac{6ABC + B^3}{3A^2}\right) = \frac{5b^3}{81a^{\frac{8}{3}}}$ *&c.*

and conſequently, $\overline{a+bx^{n}}\big|^{\frac{1}{3}}$ $(= A + Bx^{n} + Cx^{2n} + \&c.)$ $= a^{\frac{1}{3}} + \frac{bx^{n}}{3a^{\frac{2}{3}}} - \frac{b^2x^{2n}}{9a^{\frac{5}{3}}} + \frac{5b^3x^{3n}}{81a^{\frac{8}{3}}} + \&c.$

And, in the ſame Manner, may the Root of any other Quantity be extracted: But as the celebrated Binomial Theorem, diſcovered by the illuſtrious Sir *Iſaac Newton*, is vaſtly more eaſy and expeditious, in raiſing Powers and extracting Roots than that, or any other, Method, I ſhall now explain the Uſes thereof; but, firſt of all, it may not be amiſs to ſhew how the Theorem itſelf, from the Principles of Fluxions, may be derived.

Let, then, $1+y$ be a Binomial whoſe firſt Term is Unity, and its ſecond Term any propoſed Quantity y; and let the Quantity to be expanded or thrown into a Series be $\overline{1+y}|^{v}$; where the Exponent v is ſuppoſed to denote any Number whatever, whole or broken, poſitive or negative.

Now it is evident that the firſt Term of the required Series muſt be Unity; becauſe when y is $=0$, the other Terms all vaniſh; and, in that Caſe, $\overline{1+y}|^{v}$ is equal to Unity. Let, therefore, $1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. be aſſumed to expreſs the true Value of the ſaid Series, or, which is the ſame, let

$\overline{1+y}|^{v}=1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. where A, B, C, D, &c. m, n, p, q, &c. denote unknown, but determinate Quantities:

Then, by taking the Fluxion of the whole Equation, (ſuppoſing y variable) we ſhall have $v\dot{y}\times\overline{1+y}|^{v-1}=m\dot{y}Ay^{m-1}+n\dot{y}By^{n-1}+p\dot{y}Cy^{p-1}+q\dot{y}Dy^{q-1}$ &c. Whence, multiplying the Sides of the two Equations, croſs-wiſe, and dividing by $\dot{y}\times\overline{1+y}|^{v-1}$, there comes out $\overline{1+y}\times\overline{mAy^{m-1}+nBy^{n-1}+pCy^{p-1}+qDy^{q-1}\ \&c.}$ $=v+vAy^{m}+vBy^{n}+vCy^{p}+vDy^{q}$ &c. which, by Reduction, is

$$\left.\begin{array}{lllll} mAy^{m-1} & +nBy^{n-1} & +pCy^{p-1} & +qDy^{q-1} & \&c. \\ \quad * & +mAy^{m} & +nBy^{n} & +pCy^{p} & \&c. \\ -v & -vAy^{m} & -vBy^{n} & -vCy^{p} & \&c. \end{array}\right\}=0$$

Now, ſince we are at Liberty to take the Exponents of y what we will, ſo as to anſwer the Conditions of the Equation, or ſo that all the Terms here put down may mutually deſtroy each other; let them, therefore, be ſo taken that the Terms themſelves may be homologous, that is, let $m-1=0$, $n-1=m$, $p-1=n$, $q-1=p$, *&c.* Then, m being $=1$, $n=2$, $p=3$, $q=4$, *&c.* if theſe ſeveral Values be ſubſtituted above, the Equation itſelf will become

$$\left.\begin{array}{l} A+2By+3Cy^2+4Dy^3+\ \&c. \\ \ *\ +Ay\ +2By^2+3Cy^3\quad \&c. \\ -v-vAy-vBy^2-vCy^3\quad \&c. \end{array}\right\}=0$$

Where, taking $A-v=0$, $2B+A-vA=0$, $3C+2B-vB=0$, $4D+3C-vC=0$, *&c.* ſo that every Column of homologous Terms (and, conſequently, the whole Expreſſion) may vaniſh, we alſo get $A=v$; B $\left(=\frac{vA-A}{2}=\frac{A\times\overline{v-1}}{2}\right)=\frac{v\times\overline{v-1}}{2}$; C $\left(=\frac{vB-2B}{3}=B\times\frac{\overline{v-2}}{3}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}$; D $\left(=\frac{vC-3C}{4}=C\times\frac{\overline{v-3}}{4}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, *&c. &c.*

Whence, by writing theſe Values, with thoſe of m, n, p, q, *&c.* in the Series $1+Ay^m+By^n+Cy^p$ *&c.* firſt aſſumed, we, at length, find $\overline{1+y}|^v=1+vy+\frac{v}{1}\times\frac{v-1}{2}\times y^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times y^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times y^4+$ *&c.* which was to be inveſtigated.

From the Series here brought out, any Power or Root, of any other compound Quantity, whether Binomial, Trinomial, *&c.* is eaſily deduced: For, if p be put to repreſent the firſt Term of any ſuch Quantity, and Q the Quotient of the reſt of the Terms divided by the firſt; then the Quantity itſelf will be expreſſed by $P+PQ$ or $P\times\overline{1+Q}$, and the v Power thereof by $P^v\times\overline{1+Q}|^v$ which therefore is equal to

$$P^v\times\overline{1+vQ+\frac{v}{1}\times\frac{v-1}{2}\times Q^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times}$$

$$\overline{Q^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times Q^4+\&c.},$$ by what is juſt now determined.

But when v is a Fraction, as in the Notation of Roots, the Theorem here given will be render'd ſomewhat more commodious for Practice, if, inſtead of v, a Fraction as $\frac{m}{n}$ be ſubſtituted; by which means it will

become $$P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}=P^{\frac{m}{n}}\times\overline{1+\frac{m}{n}Q+\frac{m}{n}\times}$$

$$\overline{\frac{m-n}{2n}Q^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^3+\frac{m}{n}\times\frac{m-n}{2n}\times}$$

$$\overline{\frac{m-2n}{3n}\times\frac{m-3n}{4n}Q^4+\&c.}$$ whoſe Uſe, in converting radical Quantities into Infinite Serieſes will appear from the following Examples.

EXAMPLE VII.

100. *Wherein it is propoſed to extract the Square Root of a^2+x^2, in an Infinite Series.*

Here the Quantity to be expanded being $\overline{a^2+x^2}|^{\frac{1}{2}}$, or $\overline{aa}|^{\frac{1}{2}}\times\overline{1+\frac{xx}{aa}}|^{\frac{1}{2}}$, by comparing it with the general Form, $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}$, we have $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=1$, and, $n=2$: Whence, by ſubſtituting theſe Values in the laſt general Equation, we get

$$\overline{a^2+x^2}|^{\frac{1}{2}} = a \times \overline{1 + \tfrac{1}{2} \times \frac{x^2}{a^2} + \tfrac{1}{2} \times -\tfrac{1}{4} \times \frac{x^4}{a^4} + \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times \frac{x^6}{a^6}} + \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times -\tfrac{5}{8} \times \frac{x^8}{a^8} + \&c. = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\ \&c.$$

Which Series agrees exactly with thoſe found in *Art.* 97. and 98. by different Methods.

EXAMPLE VIII.

101. *Let it be required to extract the Cube-Root of b^3-y^3, in an Infinite Series.*

Here by comparing $\overline{b^3}|^{\frac{1}{3}} \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}$ $\left(= \overline{b^3-y^3}|^{\frac{1}{3}}\right)$ with $P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$, it will be $P=b^3$, $Q=-\frac{y^3}{b^3}$, $m=1$, and $n=3$: Therefore, by Subſtitution, we get

$$\overline{b^3-y^3}|^{\frac{1}{3}} \left(= b \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}\right) = b \times \overline{1 + \tfrac{1}{3} \times -\frac{y^3}{b^3} + \tfrac{1}{3} \times -\tfrac{2}{6} \times \frac{y^6}{b^6} + \tfrac{1}{3} \times \tfrac{2}{6} \times -\tfrac{5}{9} \times -\frac{y^9}{b^9} + \tfrac{1}{3} \times -\tfrac{2}{6} \times -\tfrac{5}{9} \times -\tfrac{8}{12} \times \frac{y^{12}}{b^{12}}} + \&c. = b - \frac{y^3}{3b^2} - \frac{y^6}{9b^5} - \frac{5y^9}{81b^8} - \frac{10y^{12}}{243b^{11}}\ \&c.$$

EXAMPLE IX.

102. *Let the Quantity to be converted into an Infinite Series be* $\frac{a}{\sqrt{ax-xx}}$.

In this Caſe the given Quantity being firſt transformed to $\sqrt{\frac{a}{x}} \times \overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ and $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ afterwards compared with $\overline{1+Q}|^{\frac{m}{n}}$, we have $Q = -\frac{x}{a}$, $m = -1$, and $n = 2$; and therefore $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ $(= \overline{1+Q}|^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-2n}{2n}Q^2 + \&c.)$ $1 + -\frac{1}{2} \times \frac{-x}{a} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^2}{a^2} + -\frac{1}{2} \times -\frac{3}{4} \times -\frac{5}{6} \times \frac{-x^3}{a^3}$ &c. $= 1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} + \&c.$ Which therefore, multiplied by $\sqrt{\frac{a}{x}}$, gives $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}}{8a^{\frac{3}{2}}} + \frac{5x^{\frac{5}{2}}}{16a^{\frac{5}{2}}} + \frac{35x^{\frac{7}{2}}}{128a^{\frac{7}{2}}} + \&c. = \frac{a}{\sqrt{ax-xx}}$, the Quantity propoſed.

103. It may not be improper to obſerve here, that, when both the Terms of the propoſed Quantity are affirmative, and its Exponent alſo affirmative and leſs than Unity, the two firſt Terms of the equal Series will be poſitive, and the reſt negative and poſitive, alternately; but if only the firſt Term of the Binomial be affirmative, all the Terms of the Series, after the firſt, will be negative: Moreover, if the Exponent of

the given Quantity be negative, and both the Terms affirmative, the Signs will change alternately; but if only the firſt be affirmative, all the Terms of the equal Series will be poſitive.

EXAMPLE X.

104. *Let the Quantity propoſed be the Trinomial*

$$\overline{x^3+2x^4+3x^5}\big|^{\frac{1}{3}}.$$

Here, by dividing the reſt of the Terms by the firſt, *&c.* our given Quantity is reduced to $\overline{x^3}\big|^{\frac{1}{3}} \times \overline{1+2x+3x^2}\big|^{\frac{1}{3}}$. Therefore, in this Caſe $P=x^3$, $Q=2x+3x^2$, $m=1$, and $n=3$: Whence (by Subſtitution) $\overline{x^3+2x^4+3x^5}\big|^{\frac{1}{3}} = x \times \overline{1+\frac{1}{3}\times \overline{2x+3x^2}\big| + \frac{1}{3}\times -\frac{2}{6}\times \overline{2x+3x^2}\big|^{2} + \frac{1}{3}\times -\frac{2}{6}\times -\frac{5}{9}\times \overline{2x+3x^2}\big|^{3}\ \&c.} =$ $x \times \overline{1+\frac{2x+3x^2}{3}-\frac{\overline{2x+3x^2}\big|^2}{9}+\frac{5\times\overline{2x+3x^2}\big|^3}{81}\ \&c.}$

Which, reduced to ſimple Terms, is $= x + \frac{2x^2}{3} + \frac{5x^3}{9} - \frac{68x^4}{81}$ *&c.*

105. When the propoſed Expreſſion conſiſts of a rational, multiply'd by an irrational, Quantity, the Series anſwering to the irrational one muſt be firſt found, and afterwards multiply'd by the rational Quantity: But, if two, or more, compound irrational Quantities are to be drawn into each other, then take the Series anſwering to each Quantity, ſeparately, and multiply them together; obſerving, always, to neglect all ſuch Terms whoſe Indices would exceed that of the laſt, or higheſt,

Term, which the Series ſought is propoſed to be continued to.

EXAMPLE XI.

106. *Let the Quantity propoſed be* $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}}$

Firſt we have $\overline{1-x}|^{\frac{1}{10}} = 1 - \frac{x}{10} - \frac{9x^2}{10 \times 20} - \frac{9 \times 19x^3}{10 \times 20 \times 30} - \frac{9 \times 19 \times 29x^4}{10 \times 20 \times 30 \times 40}$ —&c. Which, multiply'd by $1+x$, produces $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}} = 1 + \frac{9x}{10} - \frac{29x^2}{10.20} - \frac{9.49x^3}{10.20.30} - \frac{9.19.69x^4}{10.20.30.40}$ &c. $= 1 + \frac{9x}{10} - \frac{29x^2}{200} - \frac{147x^3}{2000} - \frac{3933x^4}{80000}$ — &c.

EXAMPLE XII.

107. *Where the Quantity to be expreſſed in an Infinite Series is* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, or $\overline{a^2-x^2}|^{\frac{1}{2}} \times \overline{c^2-x^2}|^{-\frac{1}{2}}$.

Here we have, $\overline{a^2-x^2}|^{\frac{1}{2}}$ ($a \times \overline{1 - \frac{xx}{aa}}|^{\frac{1}{2}} = a \times$
$\overline{1 + \frac{1}{2} \times -\frac{x^2}{a^2} + \frac{1}{2} \times -\frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times -\frac{x^6}{a^6}}$
$\overline{+ \text{\&c.}} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

And $\overline{c^2 - x^2}\big|^{-\frac{1}{2}}$ $\left(= c^{-1} \times \overline{1 - \frac{xx}{cc}}\big|^{-\frac{1}{2}}\right.$ $= c^{-1} \times \overline{1 + \frac{-1}{2} \times -\frac{x^2}{c^2} + \frac{-1}{2} \times \frac{-3}{4} \times \frac{x^4}{c^4} + \&c.} = \frac{1}{c} + \frac{x^2}{2c^3} + \frac{3x^4}{8c^5} + \frac{5x^6}{16c^7}$ &c. Whence, multiplying theſe two Values, one by the other, we get

$$\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 + \&c.$$

for the four firſt Terms of the Series ſought.

EXAMPLE XIII.

108. *Let the Quantity to be expanded be the Multinomial, or infinite Series,* $\overline{x^{p} + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}\big|^{v}$; *whoſe Exponent v denotes any Number whatever, whole or broken, poſitive or negative.*

Here, dividing by the firſt Term, the given Quantity is transformed to $x^{pv} \times \overline{1 + ax^{n} + bx^{2n} + cx^{3n} + dx^{4n} + \&c.}\big|^{v}$; which, if $ax^{n} + bx^{2n} + cx^{3n}$ &c. be put $= y$, will become $x^{pv} \times \overline{1 + y}\big|^{v}$; which laſt Expreſſion (by *Art.* 99.) is $= x^{pv} \times \overline{1 + vy + \frac{v}{1} \times \frac{v-1}{2} \times y^2 + \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3} \times y^3 + \&c.}$ Whence (for Brevity ſake) putting $A = v$, $B = \frac{v}{1} \times \frac{v-1}{2}$, $C = \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = \frac{v}{1} \times$

$\frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. and ſubſtituting for y, there comes out $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}\rceil^v =$ $x^{pv} + \mathrm{A}ax^{pv+n} + \overline{\mathrm{A}b + \mathrm{B}a^2} \times x^{pv+2n} +$ $\overline{\mathrm{A}c + 2\mathrm{B}ab + \mathrm{C}a^3} \times x^{pv+3n} + \overline{\mathrm{A}d + 2\mathrm{B}ac + \mathrm{B}b^2 + 3\mathrm{C}a^2b + \mathrm{D}a^4}$ $\times x^{pv+4n} + \overline{\mathrm{A}e + 2\mathrm{B}ad + 2\mathrm{B}bc + 3\mathrm{C}a^2c + 3\mathrm{C}ab^2 + 4\mathrm{D}a^3b}$ $\overline{+ \mathrm{E}a^5} \times x^{pv+5n} +$ &c.

EXAMPLE XIV.

109. *To extract the Square Root of $a^2 - x^2$, and from thence to determine the Fluent of $\dot{x}\sqrt{a^2 - x^2}$, in an Infinite Series.*

By proceeding as in the foregoing Examples, the Value of $\sqrt{a^2 - x^2}$ in an Infinite Series will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} -$ &c. Which multiplied by $\dot{x}$ gives $\dot{x}\sqrt{a^2 - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ &c. Whoſe Fluent therefore (by *Art.* 77.) is $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5}\ \frac{5x}{1152a^7} -$ &c. Which was to be determined.

EXAMPLE XV.

110. *Let it be required to approximate the Fluent of*

$$\frac{\overline{a^2-x^2}|^{\frac{1}{2}} \times x^n \dot{x}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$$ *in an Infinite Series.*

It appears, from Example 12, that the Value of $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, expressed in a Series, is $\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 +$ &c. Which Value being therefore multiplied by $x^n \dot{x}$, and the Fluent taken (by the common Method) we get $\frac{ax^{n+1}}{\overline{n+1} \times c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times \frac{x^{n+3}}{n+3} + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times \frac{x^{n+5}}{n+5} + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times \frac{x^{n+7}}{n+7} +$ &c.

EXAMPLE XVI.

111. *Wherein it is propoſed to approximate the Fluent of* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v \times x^{m-1}\dot{x}$ *in a Series.*

Here, if A be put $= v$, $B = v \times \frac{v-1}{2}$, $C = v \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = v \times \frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. the Quantity $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n}\ \&c.}|^v$ expanded, will be $= x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} + x^{pv+2n} + \overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+4n}$ + &c. as appears from *Art.* 108. Therefore this Expreſſion being multiplied by $x^{m-1}\dot{x}$, and the Fluent taken (as uſual) we ſhall have

$$\frac{x^{pv+m}}{pv+m} + \frac{Aax^{pv+m+n}}{pv+m+n} + \frac{\overline{Ab+Ba^2} \times x^{pv+m+2n}}{pv+m+2n} + \frac{\overline{Ac + 2Bab + Ca^3} \times x^{pv+m+3n}}{pv + m + 3n} + \frac{\overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+m+4n}}{pv + m + 4n} + \&c.$$

for the Quantity propoſed to be found.

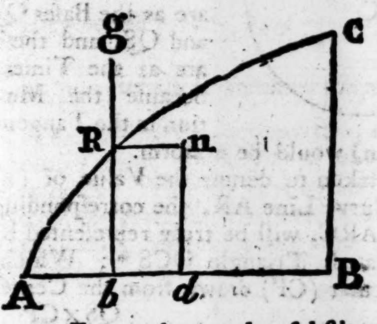
SECTION VII.

Of the Uſe of Fluxions in finding the Areas of Curves.

CASE I.

112. LET ARC *be a Curve of any Kind whoſe Ordinates are perpendicular to an Axis* AB.

Imagine a Right-line *bRg* (perpendicular to AB) to move parallel to itſelf from A towards B; and let the Celerity thereof, or the Fluxion of the Abſciſſa A*b*, in any propoſed Poſition of that Line, be denoted by *bd*: Then it will appear, *from Art.* 4. that the Rectangle (*bn*) under *bd* and the Ordinate *b*R, will expreſs the correſponding Fluxion of the generated Area *ab*R: Which Fluxion, if $Ab=x$, and $bR=y$, will therefore be $=y\dot{x}$: From whence, by ſubſtituting for y or $\dot{x}$ (according to the Equation of the Curve) and taking the Fluent, the Area itſelf will become known.

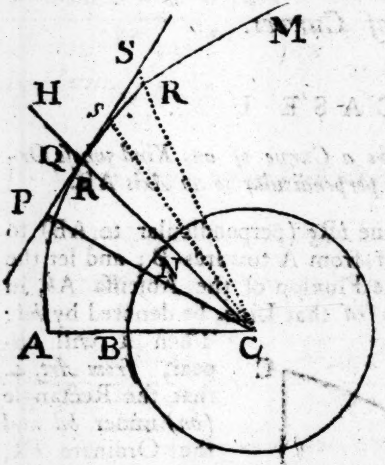


CASE II.

113. *Let* ARM *be any Curve whoſe Ordinates* CR, CR *are all referred to a Point or Center.*

Conceive a Right-line CRH to revolve about the given Center C, and let a Point R move along the ſaid Line, ſo as to trace out, or deſcribe the propoſed Curve Line ARM.

Now it is evident, that, if the Point R was to move from any Poſition Q, without changing its Direction and Velocity, it would proceed along the Tangent QS (inſtead of the Curve) and deſcribe Areas Q*s*C, QSC about the Center C, proportional to the Times of their Deſcription; becauſe thoſe Areas, or Triangles, having the ſame Altitude (CP), are as the Baſes Q*s* and QS, and theſe are as the Times, becauſe the Motion in the Tangent (upon that Suppoſition) would be uniform.



Hence, if RS be taken to denote the Value of ($\dot z$) the Fluxion of the Curve Line AR, the correſponding Fluxion of the Area ARC, will be truly repreſented by the, uniformly generated, Triangle QCS*: Which, putting the Perpendicular (CP) drawn from the Center to the Tangent, $= s$, will therefore be ($= \frac{QS \times CP}{2} = \frac{s\dot z}{2}$; from whence the Area itſelf may be determined.

But, ſince in many Caſes, the Value of $\dot z$ cannot be computed (from the Property of the Curve) without ſome Trouble, the two following Expreſſions, for the Fluxion of the Area, will commonly be found more commodious, *viz.* $\frac{sy\dot y}{2t}$ and $\frac{y^2\dot x}{2a}$; where t = RP and x = the Arch BN of a Circle, deſcribed about the Center C, at any Diſtance a ($=$ CB). Theſe Expreſſions are derived from that above, in the following Manner; *viz.* $\dot{z} : \dot{y} :: y$ (CR) $: t$ (RP) *; therefore $\dot{z} = \frac{y\dot{y}}{t}$; and *Art. 35. conſequently $\frac{s\dot{z}}{2} = \frac{sy\dot{y}}{2t}$; which is the firſt Expreſſion.

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* Art. 2 and 5.

Again, becauſe the Celerity of R in the Direction of the Tangent is denoted by $\dot{z}$, that in a Direction perpendicular to CQ (whereby the Point R revolves about the Center C) will therefore be ($= \frac{\text{CP}}{\text{CR}} \times \dot{z}$) * $=$ *Art. 35. $\frac{s\dot{z}}{y}$; which being to ($\dot{x}$) the Celerity of the Point N (about the ſame Center) as the Diſtance (or Radius) CR (y) to the Radius CN (a) we ſhall, by multiplying Extremes and Means, have $\frac{as\dot{z}}{y} = y\dot{x}$; and conſequently $\frac{s\dot{z}}{2} = \frac{y^2\dot{x}}{2a}$; which is the other Expreſſion.

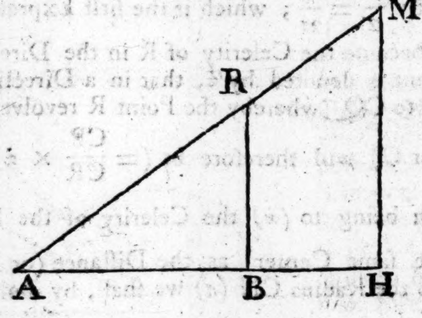
The Method of applying this, together with the preceding Forms, will appear at large from the following Examples: Wherein x, y, z, and u are all along put to denote the Abſciſſa, Ordinate, Curve-line, and the Area reſpectively, unleſs where the contrary is expreſsly ſpecified.

EXAMPLE I.

114. *Let it be propoſed to determine the Area of a right-angled Triangle* AHM.

Put the Baſe AH$=a$, the Perpendicular HM$=b$; and let AB (x) be any Portion of the Baſe, conſidered as a flowing Quantity, and let BR (y) be the Ordinate, or Perpendicular, correſponding:

Then, becauſe of the ſimilar Triangles AHM and ABR, it will be, $a : b :: x : y = \frac{bx}{a}$: Whence $y\dot{x}$



(the Fluxion of the Area ABR*) is, in this Caſe, $= \frac{bx\dot{x}}{a}$; and conſequently the Fluent thereof, or the Area itſelf $= \frac{bx^2}{2a}$ †: Which therefore, when $x=a$, and BR coincides with HM, will become $\frac{ab}{2} = \frac{\text{AH} \times \text{HM}}{2} =$ the Area of the whole Triangle AHM; which we alſo know from other Principles.

*Art. 112.

†Art. 77.

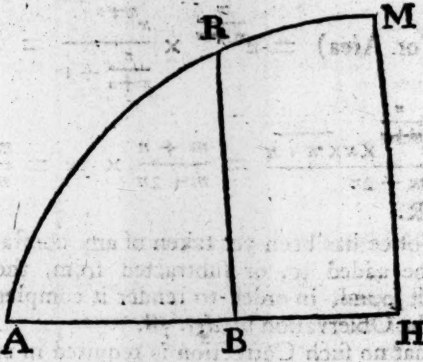
EXAMPLE II.

115. *Let the Curve* ARMH, *whoſe Area you would find, be the common Parabola.*

In which Caſe the Relation of AB (x) and BR (y) being expreſſed by $y^2 = ax$ (where a is the Parameter) we thence get $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$; and therefore $\dot{u}$ ($= y\dot{x}$ ‡) $= a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$: Whence $u = \frac{2}{3} \times a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}\, a^{\frac{1}{2}}x^{\frac{1}{2}} \times x = \frac{2}{3}yx$ (becauſe

‡Art. 112.

(becauſe $a^{\frac{1}{2}}x^{\frac{1}{2}} = y) = \frac{2}{3} \times AB \times BR$: Hence *a Parabola is* $\frac{2}{3}$ *of a Rectangle of the ſame Baſe and Altitude.*



The Area is here found in Terms of x; but it will, many times, be more eaſily brought out in Terms of y (without radical Quantities) as in the very Caſe laſt propoſed: Where x being $= \frac{y^2}{a}$, we therefore have $\dot{x} = \frac{2y\dot{y}}{a}$; and conſequently $\dot{u}\ (y\dot{x}) = \frac{2y^2\dot{y}}{a}$: Whence $u = \frac{2y^3}{3a} = \frac{2y}{3} \times \frac{y^2}{a} = \frac{2y}{3} \times x = \frac{2}{3} \times AB \times BR$; *the ſame as before.*

E X A M P L E III.

116. *Let* ARM *(ſee the preceding Figure) be a Parabola of any Kind; whereof the general Equation is* $y^{m+n} = a^m x^n$.

Therefore, by extracting the Root, or dividing each Exponent by $m+n$, we have $y = a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}}$; whence

$\dot{u}\ (y\dot{x}) = a^{\frac{m}{m+n}} \times \dot{x}x^{\frac{n}{m+n}}$; and conſequently u (the true Fluent, or Area) $= a^{\frac{m}{m+n}} \times \dfrac{x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} =$ $\dfrac{a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}} \times x \times \overline{m+n}}{m+2n} = \dfrac{m+n}{m+2n} \times yx = \dfrac{m+n}{m+2n} \times$ AB × BR.

No Notice has been yet taken of any conſtant Quantity to be added to, or ſubtracted from, the variable One, firſt found, in order to render it complete, agreeable to the Obſervation in *Art.* 78.

But that no ſuch Correction is required in any of the preceding Examples, is evident from the Nature of the Figure; becauſe, when x and y are nothing, the Area (u) ought alſo to be nothing, which it actually is according to the Equations above exhibited.

The Fluent found in the ſucceeding Example, will, however, ſtand in need of a Correction.

EXAMPLE IV.

117. *Where it is propoſed to find the Area of the Curve* ARH, *whoſe Equation is* $x^4 - a^2x^2 + a^2y^2 = 0$.

Here, the given Equation is reduced to $y = \dfrac{x \times \overline{a^2 - x^2}|^{\frac{1}{2}}}{a}$; whence $\dot{u}\ (= y\dot{x}) = \dfrac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}}{a}$: Whereof the Fluent (by the common Rule *) is —

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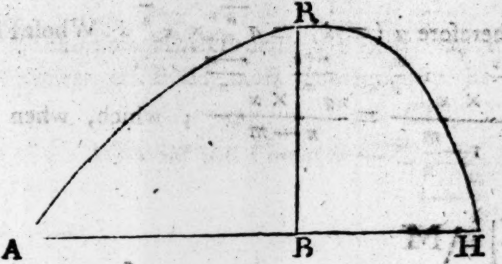
* Art. 77.



$\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$: Which, when $x=0$ and $u=0$, becomes — $\frac{a^2}{3}$; this therefore ſubtracted from $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$, leaves $\frac{a^2}{3}-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$ for the Fluent corrected, or the true Value of the Area ABR*. *Art.78.

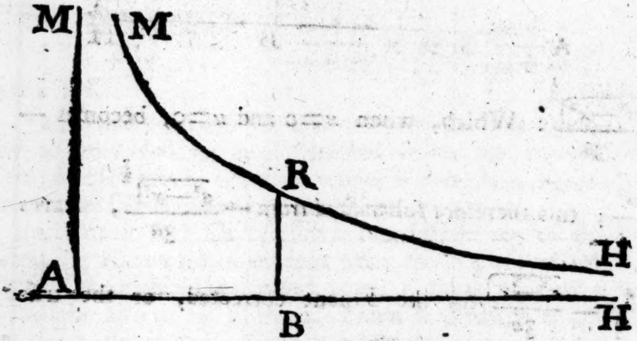
When the Ordinate BR $\left(\frac{x\sqrt{a^2-x^2}}{a}\right)$ becomes equal to Nothing, and B coincides with H, then x will become $=a=$ AH; and therefore the Area of the whole Curve ARH will be barely $=\frac{a^2}{3}=\frac{1}{3}$ AH^2

EXAMPLE V.

118. *Let it be required to determine the Area of the hyperbolical Curve whoſe Equation is* $x^m y^n = a^{m+n}$.

In this Caſe we have $y=\frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}}=a^{\frac{m+n}{n}}\times x^{\frac{-m}{n}}$;

and therefore $\dot{u}\ (=y\dot{x}) = a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}}\dot{x}$: Whoſe Fluent is $\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}} = \frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}$; which, when x is $=0$, will alſo be $=0$, if n be greater than m: Therefore, the Fluent requires no Correction in this Caſe; the Area AMRB, included between the Aſymptote AM and the Ordinate BR, being truly defined by $\left(\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}:}{n-m}\right)$ the Quantity above determined.



But, if n be leſs than m, then the Fluent, when $x=0$, will be infinite (becauſe the Index $\frac{n-m}{n}$ being negative, 0 becomes a Diviſor to $na^{\frac{m+n}{n}}$:) Whence the Area AMRB will alſo be infinite.

But, here, the Area BRH comprehended between the Ordinate, the Curve, and the Part BH of the other Aſymptote, is finite, and will be truly expounded by $\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ the ſame Quantity with its Signs changed. For the

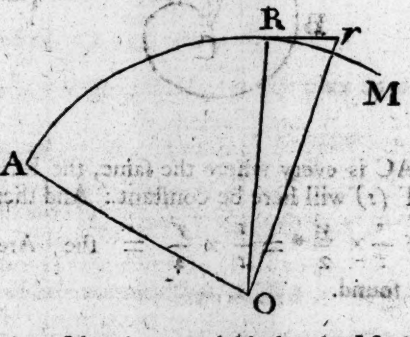
Fluxion of the Part AMRB being $a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$, that of its Supplement BRH muſt conſequently be $-a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$: Whereof the Fluent is $-\frac{a^{\frac{m+n}{n}} \times x^{\frac{1-m}{n}}}{1-\frac{m}{n}}$

$= \frac{a^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ = the Area BRH: Which wants no Correction; becauſe, when x is infinite, and the Area BRH $= 0$, the ſaid Fluent will alſo intirely vaniſh, ſeeing the Value of $x^{\frac{m-n}{n}}$ (which is a Diviſor to $a^{\frac{m+n}{n}}$) is then infinite.

EXAMPLE VI.

119. *Where let it be required to determine the Area of the circular Sector* AOR.

Then, putting the Radius AO (or OR) $= a$, the



Arch AR (conſidered as variable by the Motion of R) $= z$, and $Rr = \dot{z}$, the Fluxion of the Area will here

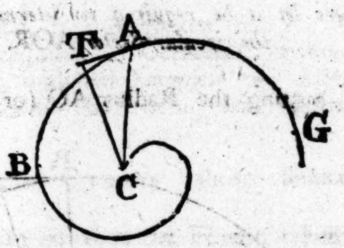
be

*Art. 113. be expreſſed by $\frac{a\dot{x}}{2}$ (= the Triangle ORr *:) Whence the Area itſelf is $= \frac{ax}{2} = AO \times \frac{1}{2}AR$: From which it appears that the Area of any Circle is expreſſed by a Rectangle under half the Circumference and half the Diameter.

EXAMPLE VII.

120. *Wherein it is propoſed to determine the Area* CBAC *of the logarithmic Spiral.*

Let the Right-line AT touch the Curve at A; upon which, from the Center C, let fall the Perpendicular CT: Then, ſince by the Nature of the Curve the

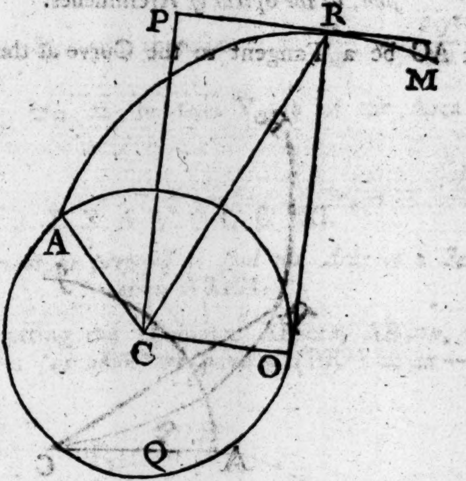


Angle TAC is every where the ſame, the Ratio of AT (t) to CT (s) will here be conſtant: And therefore the *Art. 113. Fluent of $\frac{s}{t} \times \frac{y\dot{y}}{2}$ * $= \frac{s}{t} \times \frac{y^2}{4}$ = the Area which was to be found.

EXAMPLE VIII.

121. *Let the Curve* ARM *be the Involute of a given Circle* AOQ.

In which Caſe the intercepted Part of the Tangent RP (t) being every where equal to the Radius CO (a)



of the generating Circle, we therefore have CP $(s) = \sqrt{CR^2 - RP^2}) = \sqrt{y^2 - a^2}$: Whence $\dot{u}\ (= \frac{sy\dot{y}}{2t}\ ^*) = \frac{\sqrt{y^2-a^2} \times y\dot{y}}{2a}$; and conſequently $u = \frac{\overline{y^2-a^2}|^{\frac{3}{2}}}{6a} = \frac{CP^3}{6CA}$ = the required Area ACR.

* Art. 113.

Which will alſo expreſs the Area ARO generated by the Radius of Evolution RO; becauſe, RO being = the

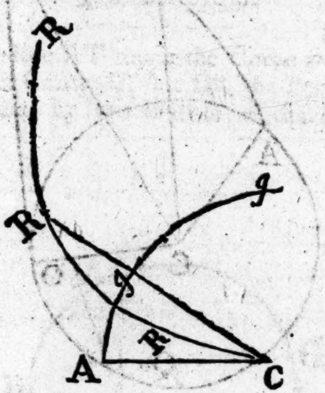
*Art.119. the Arch AO, the Sector ACO ($\frac{1}{2}$ AO × OC *) is equal to the Triangle CRO ($\frac{1}{2}$ RO × OC) which equal Quantities being ſucceſſively ſubtracted from CARO, there remains AOR=ACR.

EXAMPLE IX.

122. *Let the Curve* CRR, *whoſe Area* CR*g*C *you would find, be the Spiral of* Archimedes.

Let AC be a Tangent to the Curve at the Center C, about which Center, with any Radius AC ($=a$) ſuppoſe a Circle A*gg* to be deſcribed; then the Arch (or Abſciſſa) A*g* correſponding to any propoſed Ordinate CR, being to that Ordinate in a given, or conſtant, Ratio (ſuppoſe as *m* to *n*) we have x (A*g*) $= \frac{my}{n}$; therefore $\dot{u} = \frac{y^2\dot{x}^*}{2a} = \frac{my^2\dot{y}}{2an}$, and conſequently u *Art.113.

$= \frac{my^3}{6an} =$ the Area CRR*g*C.



EXAMPLE X.

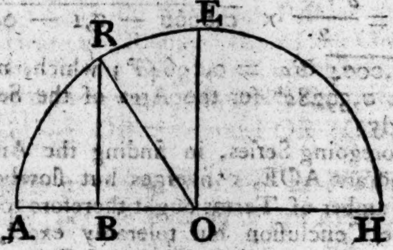
123. *Let the Equation of the Spiral* CRR *(ſee the laſt Figure) be* $x = by + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

Then, $\dot{x}$ being $= b\dot{y} + 2cy\dot{y} + 3dy^2\dot{y} + 4ey^3\dot{y} + \&c.$ we ſhall have $\dot{u}\ \left(= \frac{y^2\dot{x}}{2a}\right) = \frac{by^2\dot{y}}{2a} + \frac{2cy^3\dot{y}}{2a} + \frac{3dy^4\dot{y}}{2a} + \frac{4ey^5\dot{y}}{2a} + \&c.$ and therefore $u = \frac{by^3}{6a} + \frac{2cy^4}{8a} + \frac{3dy^5}{10a} + \frac{4ey^6}{12a}\ \&c.$ = the true Value of the Area in this Caſe.

EXAMPLE XI.

124. *Let it be propoſed to find the Area of a Semicircle* AREH.

Here, putting the Diameter $AH = a$, $AB = x$, and $BR = y$ *&c. (as uſual)* we have $y^2\ (BR^2) = ax - xx$



$(AB \times BH)$, and conſequently $\dot{u}\ (y\dot{x}) = \dot{x}\sqrt{ax - xx} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{a}}\Big|^{\frac{1}{2}}$: Which Expreſſion not being of the Kind deſcribed in *Art.* 83 and 85. that admit of Fluents in finite Terms, let it therefore be reſolved into an Infinite Series * and you will have $\dot{u} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4}}$ &c. $= a^{\frac{1}{2}} \times \overline{x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^2} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^3}}$ &c. From whence, the Fluent of every Term being taken, according to the common Method, there will come out $u = a^{\frac{1}{2}} \times \overline{\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5a} - \frac{x^{\frac{7}{2}}}{28a^2} - \frac{x^{\frac{9}{2}}}{72a^3} - \frac{5x^{\frac{11}{2}}}{704a^4}}$ &c. $= x\sqrt{ax} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{x^3}{72a^3} - \frac{5x^4}{704a^4}}$ &c. = the Area ABR. Now, when, $x = \frac{1}{2}a$, the Ordinate BR will coincide with the Radius OE; in which Caſe the Area becomes $= \frac{1}{2}a\sqrt{\frac{1}{2}aa} \times \overline{\frac{2}{3} - \frac{1}{10} - \frac{1}{112} - \frac{1}{576} - \frac{5}{11264}}$ &c. $= \frac{a^2\sqrt{\frac{1}{2}}}{2} \times \overline{0,6666 - 0,1 - 0,0089 - 0,0017 - 0,0004}$ &c. $= 0,1964a^2$; which, multiply'd by 2, gives $0,3928a^2$ for the Area of the Semi-circle AEH, nearly.

As the foregoing Series, in finding the Area of the whole Quadrant AOE, converges but ſlowly, a conſiderable Number of Terms ought therefore to be taken to have the Concluſion but tolerably exact, the five firſt Terms above collected being ſufficient to bring out no more than three Places of Figures that can be depended on. For which Reaſon it may be of Uſe to conſider, whether, by computing the Area of ſome particular Portion (ABR) of the ſaid Quadrant, that of the whole may not be deduced; where x being ſmall in

com-

* Art. 90 and 99.

compariſon of a, the Series may have ſuch a Rate of Convergency, that a ſmaller Number of Terms will be ſufficient *. *Art. 92.

Now, in order to this, it is well known that, if the Arch AR be taken $= \frac{1}{3}$ AE (or 30 Degrees) the Sine BR will be $= \frac{1}{2}$ AO; and conſequently AB $(x) =$ AO $-$ OB $=$ AO $- \sqrt{\text{OR}^2 - \text{BR}^2}$; which, if the Radius AO be expounded by Unity, (to facilitate the Operation) will be $= 0{,}1339746$ very nearly: This therefore, with the Value of a, being ſubſtituted in the forementioned Series, $\sqrt{ax^3} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \&c.}$ we have $\overline{0{,}0693505} \times \overline{0{,}6666666 - 0{,}0133975 - 0{,}0001603 - 0{,}0000042 - \&c.} = 0{,}0693505 \times 0{,}6531046 = 0{,}0452931 =$ the Area ABR: Which added to the Area OBR ($=$ OB $\times \frac{1}{2}$BR $= \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0{,}2165063$) gives 0,2617994, for the Area of the Sector AOR; the treble whereof, or 0,7853982 (becauſe AR $= \frac{1}{3}$ AE) will therefore be the Content of the whole Quadrant AOE: Which Number, found by taking four Terms of the Series only, is true to the laſt Decimal Place.

This Concluſion may be otherwiſe brought out, by finding a Series for the other Part of the Area, included between the Radius OE and the Ordinate BR; wherein the Co-ſine OB (inſtead of the verſed Sine AB) will be the converging (or variable) Quantity.

For, putting OB $= x$, and OR (OA) $= b$, we have y (BR $= \sqrt{\text{OR}^2 - \text{OB}^2} = \overline{b^2 - x^2}|^{\frac{1}{2}}$); and conſequently $(y\dot{x})$ the Fluxion of the Area OBRE * $=$ *Art. 112. $\dot{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}} = b\dot{x} - \frac{x^2\dot{x}}{2b} - \frac{x^4\dot{x}}{8b^3} - \frac{x^6\dot{x}}{16b^5} - \frac{5x^8\dot{x}}{128b^7} - \frac{7x^{10}\dot{x}}{256b^9}$ &c. Whence the Area itſelf is $= bx - \frac{x^3}{6b} - \frac{x^5}{40b^3} - \frac{x^7}{112b^5} - \frac{5x^9}{1152b^7} - \frac{7x^{11}}{2816b^9}$ &c.

Now, if x (OB) be aſſumed $= \frac{1}{2}$ AO (ſo that the Arch ER may be $= \frac{1}{3}$ AE) and the Value of b (AO) be expounded by Unity, we ſhall have

$$x^3 \left(= x \times x^2 = ,5 \times \tfrac{1}{4} = \frac{,5}{4}\right) = ,125$$

$$x^5 \left(= x^3 \times x^2 = \frac{,125}{4}\right) = ,03125$$

$$x^7 \left(= x^5 \times x^2 = \frac{,03125}{4}\right) = ,0078125$$

$$x^9 \left(= x^7 \times x^2 = \frac{x^7}{4}\right) = ,0019531 +$$

$$x^{11} \left(= x^9 \times x^2 = \frac{x^9}{4}\right) = ,0004883 -$$

&c.

Which Values of the Powers of x being reſpectively divided by 6, 40, 112, 1152, 2816, *&c.* there will reſult 0,5000000 — 0,0208333 — 0,0007812 — 0,0000698 — 0,0000085 — 0,0000012 — 0,0000002 *&c.* = 0,4783057, for the Area OBRE in the forementioned Circumſtance, when OB $= \frac{1}{2}$ OA: From which, deducting the Triangle OBR $(= \sqrt{\tfrac{3}{4}} \times \tfrac{1}{2} = 0,2165063)$ the Remainder ,2617994 will conſequently be the Area of the Sector EOR; the treble whereof (becauſe ER is, *here*, $= \frac{1}{3}$ AE) will give the Area of the whole Quadrant, 0,7853982; *as before.*

EXAMPLE XII.

125. *Let the Curve, whoſe Area you would find, be the Ciſſoid of Diocles; whereof the Equation is* $y^2 = \frac{x^3}{a - x}$.

*Art. 112. — Here we have $\dot{u}$ $(y\dot{x}\,^*)$ $= \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a - x}} = \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}} \times \overline{1 - \frac{x}{a}}\,^{\frac{1}{2}}}$

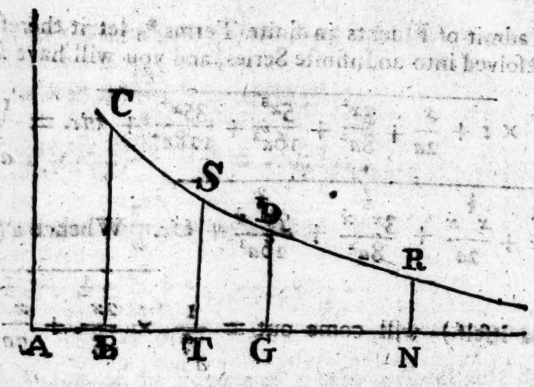
$= \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 - \frac{x}{a}}\Big|^{-\frac{1}{2}}$: Which, being none of the Kind that admit of Fluents in finite Terms *, let it therefore be reſolved into an Infinite Series, and you will have $\dot{u} =$ $\frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4}} + \&c. = \frac{1}{a^{\frac{1}{2}}} \times$ $\overline{x^{\frac{3}{2}}\dot{x} + \frac{x^{\frac{5}{2}}\dot{x}}{2a} + \frac{3x^{\frac{7}{2}}\dot{x}}{8a^2} + \frac{5x^{\frac{9}{2}}\dot{x}}{16a^3}} + \&c.$ Whence u (the Area itſelf) will come out $= \frac{1}{a^{\frac{1}{2}}} \times \overline{\frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{7}{2}}}{7a} +}$ $\overline{\frac{x^{\frac{9}{2}}}{12a^2} + \frac{5x^{\frac{11}{2}}}{88a^3}} + \&c. = x^2\sqrt{\frac{x}{a}} \times \overline{\frac{2}{5} + \frac{x}{7a} + \frac{x^2}{12a^2} +}$ $\overline{\frac{5x^3}{88a^3}} + \&c.$

*Art. 83. and 85.

EXAMPLE XIII.

126. *Let the propoſed Curve* CSDR *be of ſuch a Nature, that (ſuppoſing* AB *Unity) the Sum of the Areas* CSTBC *and* CDGBC *anſwering to any two propoſed Abſciſſas* AT *and* AG, *ſhall be equal to the Area* CRNBC *whoſe correſponding Abſciſſa* AN *drawn into* AB *is equal to,* AT × AG, *the Product of the Meaſures of the two former Abſciſſas.*

Firſt, in order to determine the Equation of the Curve, (which muſt be known before the Area can be found) let the Ordinates GD and NR move parallel to themſelves towards HF; and, then, having put $GD = y$, $NR = z$, NR $= z$, AT $= a$, AG $= s$, and AN $= u$, the Fluxion of the Area CDGB will be repreſented by $y\dot{s}$, and that



*Art.112. of the Area CRNB by $z\dot{u}$ *: Which two Expreſſions muſt, by the Nature of the Problem, be equal to each other; becauſe the latter Area CRNB exceeds the former CDGB by the Area CSTB, which is here conſidered as a conſtant Quantity; and it is evident that two Expreſſions, that differ only by a conſtant Quantity, muſt always have equal Fluxions.

Since, therefore $y\dot{s}$ is $= z\dot{u}$, and $u = as$, *by Hypotheſis*, it follows that $\dot{u} = a\dot{s}$, and that the firſt Equation (by ſubſtituting for $\dot{u}$) will become $y\dot{s} = az\dot{s}$, or $y = az$, or laſtly $ys = zas$, that is, GD × AG = NR × AN: Therefore GD : NR :: AN : AG; whence it appears that every Ordinate of the Curve is reciprocally as its correſponding Abſciſſa.

Now, to find the Area of the Curve ſo determined, put BC $= b$, and BG $= x$: Then, ſince AG $(1 + x)$: AB (1) :: BC (b) : GD (y) we have $y = \frac{b}{1+x}$, and conſequently $\dot{u}\ (= y\dot{x}) = \frac{b\dot{x}}{1+x} = b \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} - \&c.}$ Whence, BGDC, the Area itſelf will be $= b \times x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ *&c.* Which was to be found.

It may here be obſerved that the Areas of the Spaces above mentioned, are analogous to, and have the very ſame Properties as *Logarithms*; and that thoſe Spaces, or Logarithms, may be of different Forms or Values, according as you take the Value of the firſt Ordinate BC, which may be aſſumed at Pleaſure: Thus, if BC be taken = AB = Unity, the Curve will become an equilateral Hyperbola whoſe Center is A (becauſe then AG × GD = AB^2) and in that Caſe they are called hyperbolical Logarithms: But, if BC be taken =0,43429448 (ſo that the Logarithm, or the Area of the Space CDGB, anſwering to the Abſciſſa AG, when expreſſed by the Number 10, may be expounded by Unity, or AB^2) we ſhall then have the common, or *Briggian* Form of Logarithms.

From theſe Logarithms (given by the Tables) the Buſineſs of finding Fluents, is in many Caſes, very much facilitated: For, if the Fluxion given appears to agree with the Fluxion of any known Logarithmic Expreſſion, its Fluent may, it is evident, be had by the Tables, ready calculated, without the Trouble of an Infinite Series.

But, now to know what Kinds of Fluents are explicable by Means of Logarithms, it will be neceſſary to obſerve that, *the Fluxion of any hyperbolic Logarithm is always expreſſed by the Fluxion of the correſponding Number divided by that Number*: This appears from above, where *($y\dot{x}$)* the Fluxion of the Area (or Logarithm) BGDC, when BC = AB ± 1, is truly repreſented by $\frac{\dot{x}}{1+x}$; where $1 + x$ (= AG) may ſtand for any Number whatever; and $\dot{x}$ for its Fluxion.

Hence the Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expreſſed by the hyperbolical Logarithm of $x + \sqrt{x^2 \pm a^2}$: For the Fluxion of $(x + \sqrt{x^2 \pm a^2})$ the Number itſelf, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}\sqrt{x^2 \pm a^2} + x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}} \times \overline{\sqrt{x^2 \pm a^2} + x}$, this laſt Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, the very Fluxion firſt propoſed.

It alſo appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax + x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a + x + \sqrt{2ax + x^2}$: Becauſe the Fluxion of the Number $(a + x + \sqrt{2ax + x^2})$ is here $= \dot{x} + \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}} = \frac{\dot{x}}{\sqrt{2ax + xx}} \times \overline{\sqrt{2ax + xx} + a + x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax + xx}}$.

Likewiſe the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ will be repreſented by the hyperbolical Logarithm of $\frac{a+x}{a-x}$: Becauſe, the Fluxion of $\frac{a+x}{a-x}$, being $\frac{\dot{x} \times \overline{a-x} + \dot{x} \times \overline{a+x}}{\overline{a-x}|^2} = \frac{2a\dot{x}}{\overline{a-x}|^2}$, if the ſame be therefore divided by $\frac{a+x}{a-x}$, we ſhall have $\frac{2a\dot{x}}{\overline{a-x}|^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{\overline{a-x} \times \overline{a+x}} = \frac{2ax}{a^2 - x^2}$.

Laſtly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$ will be denoted by the hyperbolical Logarithm of $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2\pm x^2}} \times$

$$\frac{\overline{a+\sqrt{a^2\pm x^2}}}{\overline{a+\sqrt{a^2\pm x^2}}\big|^2} \mp \frac{x\dot{x}}{\sqrt{a^2\pm x^2}} \times \frac{\overline{a-\sqrt{a^2\pm x^2}}}{\overline{a+\sqrt{a^2\pm x^2}}\big|^2} =$$

$\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\big|^2}$; which divided by $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\big|^2} \times$

$$\frac{a+\sqrt{a^2\pm x^2}}{a-\sqrt{a^2\pm x^2}} = \frac{\pm 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\times\overline{a-\sqrt{a^2\pm x^2}}}$$

$= \frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\mp x^2} = \frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$, the Fluxion propoſed.

Theſe four are the principal Forms of Fluxions; whoſe Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occaſion, may be eaſily ſupply'd by a Table of the common Form: For, ſince the hyperbolical Logarithm of any Number is to the common Logarithm of the ſame Number, in the conſtant Ratio of Unity to 0,43429448 (as appears from above) it follows that if any common Logarithm be, either, divided by 0,43429448, or multiply'd by its Reciprocal 2,30258509, you will thence obtain the hyperbolical Logarithm correſponding.

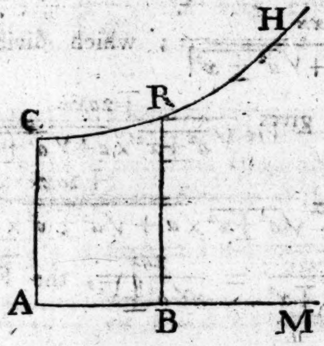
EX-

EXAMPLE XIV.

127. *Let it be required to determine the Area of the Curve; whoſe Equation is* $a^2y - x^2y - a^3 = 0$.

*Art. 112. In which Caſe y being $= \frac{a^3}{a^2 - x^2}$, we have $\dot{u}$ $(= y\dot{x})$ *

$$= \frac{a^3\dot{x}}{a^2 - x^2} = a\dot{x} + \frac{x^2\dot{x}}{a} + \frac{x^4\dot{x}}{a^3} + \frac{x^6\dot{x}}{a^5} + \frac{x^8\dot{x}}{a^7} + \&c.$$



Whence $u = ax + \frac{x^3}{3a} + \frac{x^5}{5a^3} + \frac{x^7}{7a^5} + \frac{x^9}{9a^7} + \&c.$ = the Area ſought.

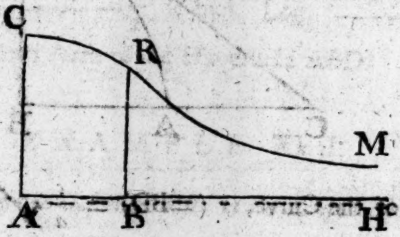
But the ſame Area (or Fluent) may be found without an Infinite Series, by Means of a Table of Logarithms, agreeable to the Obſervations in the laſt Article: For, ſince it there appears that the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is truly expreſſed by the hyperbolic Logarithm of $\frac{a + x}{a - x}$, it follows that that of $\frac{a^3\dot{x}}{a^2 - x^2}$ $\left(= \frac{2a\dot{x}}{a^2 - x^2} \times \frac{1}{2}a^2\right)$ will be expreſſed by the ſame Logarithm multiply'd by $\frac{1}{2}a^2$. Thus, for Example ſake, let a (=AC) be taken = 10, and x (= AB) = 5; then will $\frac{a+x}{a-x} = 3$; whoſe Logarithm taken from the common Tables is 0,4771213; which multiply'd by the *Modulus* 2,30258509 (ſee the laſt Article) gives 1,0986123 for the hyperbolical Logarithm of $\frac{a+x}{a-x}$; and this again multiply'd by 50 ($\frac{1}{2}a^2$) produces 54,930614 for the true Value of the Area ABRC, in the aforeſaid Circumſtance, when AC = 10, and AB = 5.

EXAMPLE XV.

128. *Where the propoſed Curve, is that, whoſe Equation is* $a^2y^2 + x^2y^2 = a^4$.

Here, by reducing the given Equation, we get $y = \frac{a^2}{\sqrt{a^2+x^2}}$: Therefore $y\dot{x} = \frac{a^2\dot{x}}{\sqrt{a^2+x^2}} =$ *. *Art. 11.

Whence, the Fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being = hyperb.



Log. of $x + \sqrt{a^2+x^2}$ (by *Art.* 126. that of $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ will conſequently be = the ſame Logarithm multiply'd by a^2.

But to find whether the Fluent thus determined does not need a Correction ‡, let x be taken = 0; then the ‡Art. 78.
Fluent

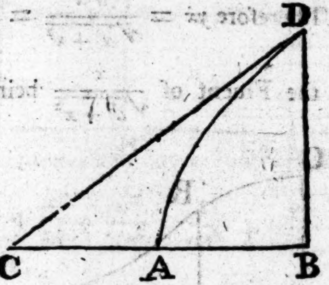
Fluent will become = hyp. Log. $a : \times a^2$: Which, therefore, muſt be ſubtracted, to have the true Value of the Area ACRB *; and then there reſults $a^2 \times$ hyp. Log. $\overline{x + \sqrt{a^2+x^2}} - a^2 \times$ hyp. Log. $a = a^2 \times$ hyp. Log. $\dfrac{x+\sqrt{a^2+x^2}}{a} = u$.

*Art. 78.

EXAMPLE XVI.

129. *Let it be propoſed to find the Area of the Hyperbola* ABD, *and alſo the Area of the hyperbolical Sector* CAD; *ſuppoſing* C *to be the Center, and* A *the principal Vertex of the Curve.*

Here, putting the Semi-tranſverſe Axis CA $= a$, the Semi-conjugate $= c$, and CB $= x$; we have, by the



Property of the Curve, y (=BD) $= \dfrac{c}{a}\sqrt{xx - aa}$; and therefore $\dot{u} = y\dot{x} = \dfrac{c\dot{x}}{a}\sqrt{x^2 - a^2}$ = the Fluxion of the Area ABD ‡

‡Art. 112.

But to find the Fluxion of the Sector CAD, it is to be obſerved, that as the ſaid Sector is = CBD − ABD $= \dfrac{xy}{2} - u$, its Fluxion will therefore be =

$\frac{x\dot{y}}{2} + \frac{y\dot{x}}{2} - \dot{u} = \frac{x\dot{y}}{2} - \frac{y\dot{x}}{2}$ (becauſe $\dot{u} = y\dot{x}$*) which, by ſubſtituting for y and $\dot{y}$, their Equals $\frac{c}{a}\sqrt{x^2 - a^2}$ and $\frac{cx\dot{x}}{a\sqrt{x^2 - a^2}}$, is at length reduced to $\frac{ac}{2} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$: Whereof the Fluent (by *Art.* 126.) is $\frac{ac}{2} \times$ hyp. Log. $x + \sqrt{x^2 - a^2}$; which corrected (by making $x = a$) will become $\frac{ac}{2} \times$ hyp. Log. $x + \sqrt{x^2 - a^2} - \frac{ac}{2} \times$ hyp. Log. $a = \frac{ac}{2} \times$ hyp. Log. $\frac{x + \sqrt{x^2 - a^2}}{a}$ = the Sector ADC: Which, ſubtracted from $\frac{cx\sqrt{x^2 - a^2}}{2a}$ $\left(= \frac{\text{BC} \times \text{BD}}{2}\right.$ = the Triangle ABD) leaves $\frac{cx\sqrt{x^2 - a^2}}{2a} - \frac{ac}{2} \times$ hyp. Log. $\frac{x + \sqrt{x^2 - a^2}}{a}$ for the required Area of the Hyperbola ABD.

*Art. 112.

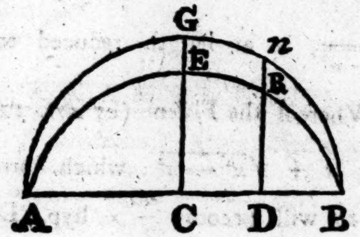
EXAMPLE XVII.

130. *Let the Curve propoſed be the Ellipſis* AEB.

Then, putting the tranſverſe Axis AB $= a$, and the Conjugate (2CE) $= c$; we ſhall, by the Property of the Curve, have y (DR) $= \frac{c}{a}\sqrt{ax - xx}$, and therefore $\dot{u}\ (y\dot{x}) = \frac{c}{a} \times \dot{x}\sqrt{ax - xx}$ = the Fluxion of the Area ARD.

But

But $\dot{x}\sqrt{ax-xx}$ is known to expreſs the Fluxion of the correſponding Segment AD*n* of the circumſcribing

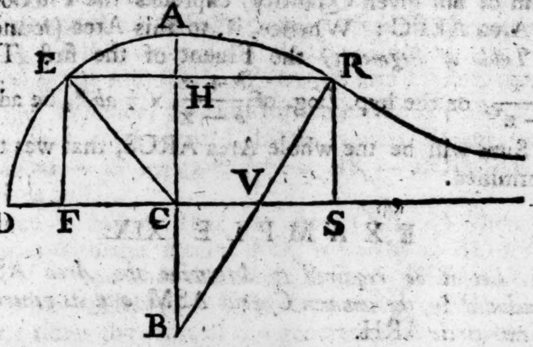


Semi-circle; whoſe Fluent is, therefore, given, by *Art.* 124; which being denoted by A, that of $\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ will, conſequently, be $=\frac{c}{a}\times A$. Hence, the Area of the Segment of an Ellipſis, is to the Area of the correſponding Segment of its circumſcribing Circle, as the leſſer Axis of the Ellipſis is to the greater; whence, it follows that the whole Ellipſis muſt be to the whole Circle in the ſame Ratio.

EXAMPLE XVIII.

131. *Let the Curve* AR &c. *whoſe Area* CARS *you would find, be the Conchoid of* Nicomedes.

Whereof the Equation (putting BC $= a$, and RV ($=$ AC) $= b$) is $x^2y^2=\overline{a+y}|^2\times\overline{b^2-y^2}$ (*Vid. Art.* 57.) Which, by Reduction, becomes $x=\frac{a\sqrt{b^2-y^2}}{y}+$



$\sqrt{b^2-y^2}$: But, to bring it down to a, *ſtill*, more ſimple Form, make $\sqrt{b^2-y^2}$ (= SV) = z; then $y =$ $\sqrt{b^2-z^2}$; whence, by Subſtitution, $x = \frac{az}{\sqrt{b^2-z^2}}$ $+ z$; and conſequently $\dot{x} =$

$$\frac{a\dot{z}\sqrt{b^2-z^2} + \frac{z\dot{z}}{\sqrt{b^2-z^2}} \times az}{b^2-z^2} + \dot{z} =$$

$$\frac{a\dot{z} \times \overline{b^2-z^2} + az^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z} = \frac{ab^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z};$$

and therefore $\dot{u}\ (y\dot{x}) = \sqrt{b^2-z^2} \times \overline{\frac{ab^2\dot{z}}{\overline{b^2-z^2} \times \sqrt{b^2-z^2}} + \dot{z}} = \frac{ab^2\dot{z}}{b^2-z^2} + \dot{z}\sqrt{b^2-z^2}$.

But now, to exhibit the Fluent hereof; upon C, as a Center, with the Radius AC *(b)* let a Quadrant of a Circle AED be deſcribed, and let RH, produced, meet the Periphery thereof in E, alſo let EF be parallel to AC, and let CE be drawn: It is evident (becauſe CE (CA) = VR and EF = RS) that CF is alſo = VS = z; and therefore, EF being $(= \sqrt{CE^2 - CF^2}) =$ $\sqrt{b^2-z^2}$, it appears that $\dot{z}\sqrt{b^2-z^2}$ (the ſecond

Term

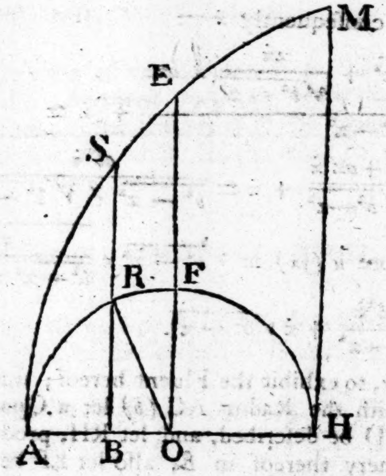
Term of our given Quantity) expreſſes the Fluxion of the Area AEFC: Whence, if to this Area (found *by the Table of Segments*) the Fluent of the firſt Term $\frac{ab^2\dot{z}}{b^2-z^2}$, or the hyp. Log. of $\frac{b+z}{b-z}$, × $\frac{1}{2}ab$ *, be added, the Sum will be the whole Area ARCS, that was to be determined.

* Art. 126.

EXAMPLE XIX.

132. *Let it be required to determine the Area* ASRA *included by the common Cycloid* ASM *and its generating Semi-circle* ARH.

Put the Radius AO (or RO) $= a$, the Sine BR $= y$, the Co-ſine OB $= x$, and the Arch AR ($=$ RS, by the Property of the Cycloid) $= z$: Then AB being $= a$



$-\,x$, its Fluxion will be $-\dot{x}$; whence (u) that of the Area ARS is $= -z\dot{x}$ *. Now to find the Fluent thereof, make $w = -zx$, ($=$ the Fluent, if z was conſtant)

* Art. 112.

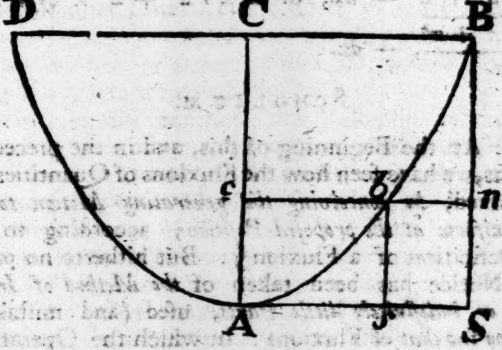
ſtant) then $\dot{w}$ being $= -z\dot{x} - x\dot{z}$*, we ſhall have *Art. 20.
$\dot{u}$ $(= -z\dot{x}) = \dot{w} + x\dot{z}$. But (*by Art.* 35.) $\dot{x}$ (AR Fluxion) : $\dot{y}$ (BR Fluxion) : : Radius : Co-ſine of the Angle ARB, or its Equal ROB : : OR (a) : OB (x): Therefore, by multiplying Extremes and Means, we get $x\dot{x} = a\dot{y}$: Whence, by Subſtitution $\dot{u}$ $(= \dot{w} + x\dot{x}) = \dot{w} + a\dot{y}$; and conſequently, by taking the Fluent, $u = w + ay = -zx + ay =$ AO × BR − BO × AR = the Area ARS.

Hence it follows that the Area (AEFA) when RB coincides with the Radius FO, is barely = AO × FO = AO^2: And that the whole Area AMHFA is truly defined by − ARH × − OH, or by ARH × OH; that is by four times the Area of the generating Semi-circle.

EXAMPLE XX.

133. *Let the Curve propoſed be the* Catenaria DAB.

Then, drawing BS and *bs* parallel to the Axis AC, and AS and *cbn* perpendicular to the ſame; and making (as uſual) A$c = x$, $cb = y$ and A$b = z$, we ſhall have, by



the Property of the Curve, $2ax + x^2 = zz$: Whence $x = \sqrt{a^2 + z^2} - a$, and $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$: From which the

Value

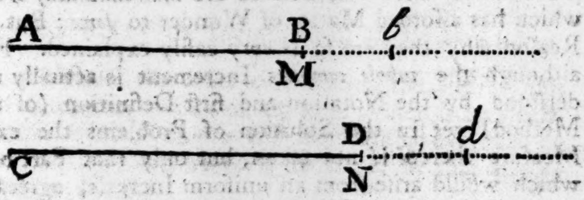
*Art. 135. Value of $\dot y$ (which in all Curves is $= \sqrt{\dot z^2 - \dot x^2}$ *) will here be found $= \sqrt{\dot z^2 - \frac{z^2\dot z^2}{a^2+z^2}} = \sqrt{\frac{a^2\dot z^2}{a^2+z^2}} = \frac{a\dot z}{\sqrt{a^2+z^2}}$; and this multiplied by $\sqrt{a^2+z^2} - a$ ($= bs$) gives $a\dot z - \frac{a^2\dot z}{\sqrt{a^2+z^2}}$ (= the Rectangle S*b*)
†Art. 112. = the Fluxion of the Area A*sb* †. From whence, by taking the Fluent, the Area itſelf is found $= az, - a^2$
‡Art. 126. $\times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ ‡: Which therefore deducted from the Rectangle *sc* ($= yx = y\sqrt{a^2+z^2} - ay$) leaves $y\sqrt{a^2+z^2} - ay - az, + a^2 \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ for the required Area A*bc*. But, ſince $\dot y = \frac{a\dot z}{\sqrt{a^2+z^2}}$ we have $y = a \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$; whence, by Subſtitution, the Area, at laſt comes out $= y\sqrt{a^2+z^2} - az$, or $= a\sqrt{a^2+z^2} \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}, - az$.

SCHOLIUM.

134. At the Beginning of this, and in the preceding Sections, we have ſeen how the Fluxions of Quantities are determined, *by conceiving the generating Motion to become uniform at the propoſed Poſition*; according to the
§ Art. 2. true Definition of a Fluxion §: But hitherto no particular Notice has been taken of *the Method of Increments, or indefinitely little Parts*, uſed (and miſtaken) by *many* for *that* of Fluxions: In which the Operations are, for the general Part, exactly the ſame; and which (tho' leſs accurate) may be applied to good Purpoſe in finding the Fluxions themſelves, in many Caſes. For which Reaſons it may not be improper to add here a

a few Lines on that Head, to ſhew the Beginner how the two Methods differ from each other; eſpecially as we ſhall be enabled, from thence, to draw out ſome Concluſions that will be of Uſe in the enſuing Part of the Work.

It hath been frequently inculcated in the foregoing Pages, that *the Fluxions of Quantities are always meaſured by how much the Quantities themſelves* would be *uniformly augmented in a given Time.* Therefore, if two Quantities or Lines, AB and CD be generated together, by the uniform (or equable) Motion of two Points B and D, it follows, that any two Spaces B*b* and D*d* *actually* gone over (whereby AB and CD are augmented) in the ſame time, will truly expreſs the Fluxions of the generated Lines AB and CD: Whence it appears that the Increments (or Spaces actually gone over) and the Fluxions are the ſame in this Caſe, where the generating Velocities are equable.



But if, on the contrary, the Velocities of the two Points, in generating the Increments M*b* and N*d*, be ſuppoſed either to increaſe, or to decreaſe, the Lines or Increments ſo generated will, it is plain, no longer expreſs the Fluxions of AB and CD; being greater, or leſs than the Spaces that *might be uniformly* deſcribed, in the ſame Time, with the Velocities at M and N.

If, indeed, thoſe Increments, and the Time of their Deſcription, be taken ſo exceeding ſmall that the Motion of the Points during that Time may be conſidered as equable, the Ratio of the ſaid Increments, will then expreſs that of the Fluxions, or be as the Velocity at M to that at N, indefinitely near; but cannot be conceived to be *ſtrictly ſo*; unleſs, perhaps, in certain particular Caſes.

Hence we ſee that the *Differential Method*, which proceeds upon theſe indefinitely little Increments (actually generated) as we do upon Fluxions (or the Spaces that *might be uniformly* generated) differs little, or nothing, from the Method of Fluxions, except in the Manner of Conception, and in Point of Accuracy, wherein it appears defective: And yet it is very certain the Concluſions this Way derived are *mathematically* true; which has afforded Matter of Wonder to *ſome*: But the Reaſon why they are ſo is very eaſily explained. For, although the *whole complete* Increment is actually underſtood by the Notation and firſt Definition (of this Method) yet in the Solution of Problems the exact Meaſure thereof is not taken, but only that Part of it which would ariſe from an uniform Increaſe, agreeable to the Notion of a Fluxion; which admits of a ſtrict Demonſtration: But, after all, the *Differential Method* has one Advantage above that of Fluxions, which is, we are not there obliged to introduce the Properties of Motion. Since we reaſon upon the Increments themſelves, and not upon the Manner in which they may be generated.

It has been hinted above, that, though the Increments of Quantities are not, *ſtrictly*, as the Fluxions, yet from them the Ratio of the Fluxions may be deduced; and it appears that the ſmaller thoſe Increments are taken, the nearer their Ratio will approach to that of the Fluxions. Therefore, if we can, by any Means, find the Ratio to which the ſaid Increments, by conceiving them leſs and leſs, do perpetually converge, and which they may approach, before they vaniſh, nearer than any aſſignable Difference, that Ratio (called hereafter, for Diſtinction Sake, *the Ratio limiting that of the Increments*) will be, *ſtrictly*, that of the Fluxions.

This will more particularly appear from the following Inſtances; wherein the Manner of deriving the Ratio of the Fluxions, from that of the Increments, is ſhewn.

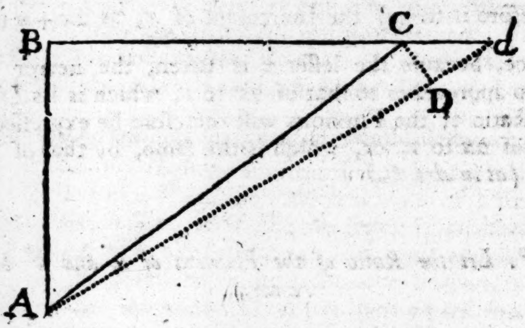
1°. *Let it be propoſed to determine the Ratio of the Fluxions of x and x^2.*

Now, if x be ſuppoſed to be augmented by any (ſmall) Quantity $\overset{\prime}{x}$, ſo as to become $x + \overset{\prime}{x}$, its Square (x^2) will be augmented to $\overline{x+\overset{\prime}{x}}^{2} = x^2 + 2x\overset{\prime}{x} + \overset{\prime}{x}\overset{\prime}{x}$; whence the Increment of x^2 will be $2x\overset{\prime}{x} + \overset{\prime}{x}\overset{\prime}{x}$; which therefore is to ($\overset{\prime}{x}$) the Increment of x, as $2x+\overset{\prime}{x}$ to 1. Hence, becauſe the leſſer $\overset{\prime}{x}$ is taken, the nearer this Ratio approaches to that of $2x$ to 1, which is its *Limit*, the Ratio of the Fluxions will therefore be expreſſed by that of $2x$ to 1, or, which is the ſame, by that of $2x\dot{x}$ to $\dot{x}$ (*as in Art.* 6.)

2°. *Let the Ratio of the Fluxions of x and x^n be required.*

Then, if x be augmented to $x+\overset{\prime}{x}$, x^n will be augmented to $\overline{x+\overset{\prime}{x}}^{n} = x^n + nx^{n-1}\overset{\prime}{x} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\overset{\prime}{x}^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\overset{\prime}{x}^3$ &c. (*Vid. Art.* 99. Whence the Increments of x and x^n will be to each other as 1 to $nx^{n-1} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\overset{\prime}{x} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\overset{\prime}{x}\overset{\prime}{x}$ &c. Where the ſmalleſt $\overset{\prime}{x}$ is taken, the nearer the Ratio will approach to that of 1 to nx^{n-1}; which appears to be its Limit: Therefore this laſt Ratio, or that of $\dot{x}$ to $nx^{n-1}\dot{x}$, is the Ratio of the Fluxions required. *(Vid. Art.* 8.*)*

3°. *Let it be propoſed to determine the Proportion of the Fluxions of the Sides* AC *and* BC, *of a right-angled, plane Triangle* ABC; *ſuppoſing the Perpendicular* AB *to remain invariable.*



If C*d* be aſſumed to repreſent any Increment of BC and D*d*, the correſponding Increment of AC (=AD) the Ratio of thoſe Increments will be, univerſally, expreſſed by that of the Sine of the Angle CD*d* to the Sine of the Angle DC*d* *(by plane Trigonometry)* and the leſs the Increments are ſuppoſed to be, the nearer will the Angle CD*d* approach to a right one, or to an Equality with B; which is its Limit: And the nearer will DC*d* approach, at the ſame time, to an Equality with BAC. Therefore *the Ratio here limiting that of the Increments* is that of the Sine of B (or Radius) to the Sine of BAC: Which alſo expreſſes that of the required Fluxions. *(Vid. Art.* 35.*)*

In the ſame way the Proportion of the Fluxions of other Kinds of algebraical and geometrical Quantities may be inveſtigated; but it will be unneceſſary to dwell longer upon this Head: I ſhall therefore only add one other Obſervation from hence (which will be of uſe hereafter) relating to the Value of an algebraic Fraction, in that particular Circumſtance when both its Numerator and Denominator become equal to Nothing, or vaniſh, at the ſame time. Which Value (it follows from above) will *be found by dividing the Fluxion of the Numerator by that of the Denominator.*

For, ſince the Value of any Fraction, in that Circumſtance, is to be looked on as *the limiting Ratio* towards which its two Terms converge, before they vaniſh, and ſeeing the Fluxions are, always, expreſſed by that Ratio, the Truth of the Rule, or Poſition, is manifeſt.

An Example, however, may not be improper:

Let therefore the Fraction $\frac{x^2-a^2}{x-a}$ be propounded, to find the Value thereof when $x=a$. In which Caſe, the true Value ſought, or the Fluxion of the Numerator divided by that of the Denominator, is $=\frac{2x\dot{x}}{\dot{x}}$ $=2x=2a$. And that this is the true Value, may be confirmed by common Diviſion, whereby the Fraction propoſed is reduced to $x+a$; whoſe Value when $x=a$, is therefore $=2a$, *the very ſame as before.*

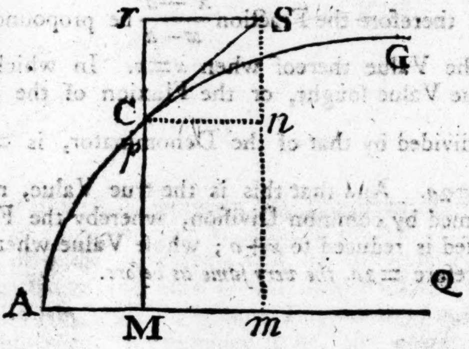
SECTION VIII.

The Uſe of Fluxions in the Rectification, or finding the Lengths, of Curves.

CASE I.

135. LET ACG *be a Curve of any Kind whoſe Ordinates are parallel to themſelves and perpendicular to the Axis* AQ.

If the Fluxion of the Abſciſſa AM be denoted by M*m*, or by C*n* (equal and parallel to M*m*) and *n*S,



equal and parallel to C*r*, be taken to repreſent the correſponding Fluxion of the Ordinate MC; then will the Diagonal CS (touching the Curve in C *) be the Line which the generating point (*p*) would deſcribe, was its Motion to become uniform at C (*Vid. Art.* 48 and 49.) which Line, therefore, truly expreſſes the Fluxion of the Space AC gone over, according to *the Definition* †.

* Art. 48 and 49.

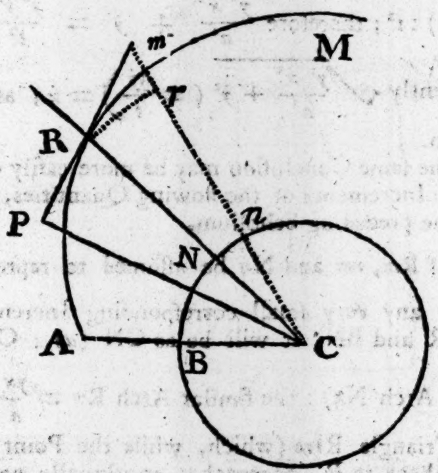
† Art. 2.

Hence, putting AM$=x$, CM$=y$, and AC$=z$, we have $\dot{z}$ ($=$CS$=\sqrt{Cn^2+Sn^2}$) $=\sqrt{\dot{x}^2+\dot{y}^2}$; from which, and the Equation of the Curve, the Value of z may be determined.

CASE II.

136. *Let all the Ordinates of the propoſed Curve* ARM *be referred to a Center* C.

Then, putting the Tangent RP (intercepted by the Perpendicular CP) $= t$, the Arch BN, of a Circle deſcribed about the Center C$=x$; the Radius CN (or CB) $=a$, &c. (*Vid. Art.* 113.) we have $\dot{z} : \dot{y} :: y$ (CR)



: t (RP*) and conſequently $\dot{z} = \frac{y\dot{y}}{t}$: From whence the Value of z will be found, if the Relation of y and t is given.

* Art. 35.

But in other Caſes it will be better to work from the following Equation, *viz.* $\dot{z} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}$. Which is thus derived.

Let the Right Line, CR, be conceived to revolve about the Center C; then ſince the Celerity of the generating Point R in a Direction perpendicular to CR is to ($\dot{x}$) the Celerity of the Point N, as CR (y) to CN (a) it will therefore be truly repreſented by $\frac{y\dot{x}}{a}$: Which being to ($\dot{y}$) the Celerity in the Direction of CR, produced, as CP (s) : RP (t) * it follows that $\frac{y^2\dot{x}^2}{a^2} : \dot{y}^2 :: s^2 : t^2$: Whence, by Compoſition, $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 : \dot{y}^2 :: s^2 + t^2\,(y^2) : t^2$; therefore $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 = \frac{y^2\dot{y}^2}{t^2}$, and conſequently $\sqrt{\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2}\left(= \frac{y\dot{y}}{t}\right) = \dot{z}$; as was to be ſhewn.

*Art. 35.

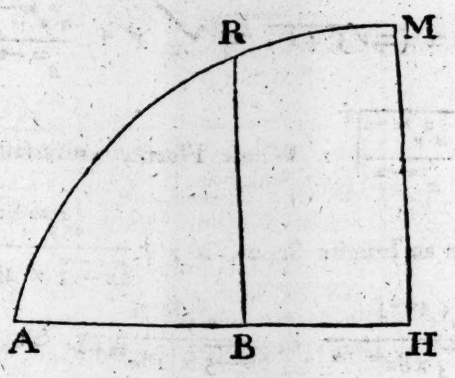
But the ſame Concluſion may be more eaſily deduced from the Increments of the flowing Quantities, according to the preceding Scholium.

For, if Rm, rm and Nn be aſſumed to repreſent ($\acute{z}$, $\acute{y}$ and $\acute{x}$) any very ſmall corresponding Increments of AR, CR and BN, it will be as CN (a) : CR (y) :: $\acute{x}$ (the Arch Nn) : the ſimilar Arch Rr = $\frac{y\acute{x}}{a}$. And, if the Triangle Rrm (which, while the Point m is returning back to R, approaches continually nearer and nearer to a Similitude with CRP) be conſidered as *rectilineal*, we ſhall alſo obtain $\acute{z}^2$ ($= \text{R}m^2 = \text{R}r^2 + rm^2$) $= \frac{y^2\acute{x}^2}{a^2} + \acute{y}^2$: Whence, by writing $\dot{z}$, $\dot{x}$ and $\dot{y}$ for $\acute{z}$, $\acute{x}$ and $\acute{y}$ (according to the Scholium) there comes out $\dot{z}^2 = \frac{y^2\dot{x}^2}{a^2} + \dot{y}^2$, *as before.*

EXAMPLE I.

137. *Let the Curve* ARM *whoſe Length is ſought, be the Semi-cubical Parabola.*

Whereof the Equation being $ax^2=y^3$, or $x=\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, we thence have $\dot{x}=\frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: Whence $\dot{z}$ $(=\sqrt{\dot{y}^2+\dot{x}^2}$*) *Art. 135.



$=\sqrt{\dot{y}^2+\frac{9y\dot{y}^2}{4a}}=\frac{\dot{y}\times\overline{4a+9y}|^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$. Whoſe Fluent (found by the common Rule) is $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$; which, corrected (by making $y=0$) becomes $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{2}}$. $-\frac{8a}{27}=z$.

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EXAMPLE II.

138. *Let the Curve propoſed be a Parabola of any (other) Kind.*

Then $x = \frac{y^n}{a^{n-1}}$ being a general Equation to all Kinds of Parabolas, we here have $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$, and therefore $\dot{z}$ $(=\sqrt{\dot{y}^2+\dot{x}^2}) = \sqrt{\dot{y}^2 + \frac{n^2y^{2n-2}\dot{y}^2}{a^{2n-2}}} = \dot{y} \times \overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}}$: Whoſe Fluent, univerſally expreſſed in an Infinite Series, is $y + \frac{n^2y^{2n-1}}{\overline{2n-1} \times 2a^{2n-2}} - \frac{n^4y^{4n-3}}{\overline{4n-3} \times 8a^{4n-4}} + \frac{n^6y^{6n-5}}{\overline{6n-5} \times 16a^{6n-6}}$, &c. $= z$.

But, when $2n-2$, the Index of y, in the given Fluxion, is either equal to Unity, or to any *aliquot Part* of it, the Fluent may be accurately had in finite Terms, *by Article* 84.

For, by putting $\frac{1}{2n-2} = v$, and $\frac{n^2}{a^{2n-2}} = c$, our Fluxion $\left(\overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y}\right)$ is, in the firſt place, reduced to $\overline{1+cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}} \times \dot{y}$: Which being compared with $\overline{a+cz^{n}}\big|^{m} \times dz^{rn-1}\dot{z}$, the general Expreſſion in the foreſaid *Article*, we have $a=1$, $z=y$, $n=\frac{1}{v}$, $m=\frac{1}{2}$, $d=1$, $\dot{z}=\dot{y}$, $rn-1=0$, or $\frac{r}{v}-1=0$; whence $r=v$, $s\ (r+m)=v+\frac{1}{2}$; and conſequently

$$\frac{d\times\overline{a+cz^{n}}\big|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\&c.}^{*}$$

$$\frac{\overline{1+cy^{\frac{1}{v}}}\Big|^{\frac{3}{2}}}{c+\frac{c}{2v}}\times\overline{y^{\frac{v-1}{v}}-\frac{\overline{v-1}\times y^{\frac{v-2}{v}}}{\overline{v-\frac{1}{2}}\times c}+\frac{\overline{v-1}\times\overline{v-2}\times y^{\frac{v-3}{v}}}{\overline{v-\frac{1}{2}}\times\overline{v-\frac{3}{2}}\times c^{2}}-\&c.}$$

$=$ the Fluent of $\overline{1+cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}}\times\dot{y}$; which was to be determined, and which will (it is plain) always terminate in v Terms, when v, or its Equal $\frac{1}{2n-2}$, is a whole poſitive Number.

If $\frac{2v+1}{2v}$ (derived from $v=\frac{1}{2n-2}$) be ſubſtituted for its Equal n, the Equation of the Curve, will be changed to $ax^{2v}=y^{2v+1}$; which, if v be expounded by 1, 2, 3, 4, &c. ſucceſſively, will become $ax^{2}=y^{3}$, $ax^{4}=y^{5}$, $ax^{6}=y^{7}$, $ax^{8}=y^{9}$ &c. reſpectively: In all which Caſes the Length of the Curve may therefore be *accurately* had from the Fluent above exhibited.

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[*] Art. 84.

Moreover, if n be aſſumed $= 2$ (or $v = \frac{1}{2}$) the general Equation, $x = \frac{y^n}{a^{n-1}}$, will then become $x = \frac{y^2}{a}$; anſwering to the common (or conical) Parabola.

And therefore in that Caſe $\dot{z}$ $\left(= \overline{1 + \frac{n^2 y^{2n-2}}{a^{2n-2}}}\right|^{\frac{1}{2}} \times \dot{y})$ is $= \dot{y}\sqrt{1 + \frac{4y^2}{a^2}} = \frac{\dot{y}\sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a} = \frac{\dot{y}\sqrt{b^2 + y^2}}{b}$ (by putting $b = \frac{1}{2}a$) $= \frac{\dot{y} \times \overline{b^2 + y^2}}{b\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2\dot{y} + y^2\dot{y}}{\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2 y\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2\dot{y}}{\sqrt{b^2 + y^2}}$: Where, the Fluent of the firſt Term (of the Fluxion ſo transformed) being $= \frac{1}{2}\sqrt{b^2y^2 + y^4}$ (or $\frac{1}{2}y\sqrt{b^2 + y^2}$ by the common Rule; and that of the ſecond Term

* Art.126. $= \frac{1}{2}b^2 \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$, * it follows that the Length of the Curve will, in this Caſe, be $= \frac{\frac{1}{2}y\sqrt{b^2 + y^2}}{b} + \frac{1}{2}b \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$.

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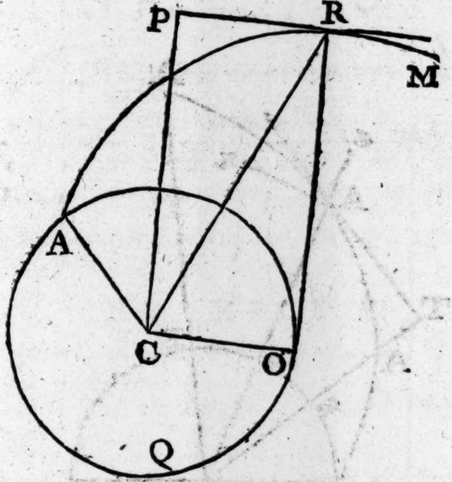
EXAMPLE III.

139. *Let the Curve propoſed be the Involute of a Circle;* whoſe Nature is ſuch, that the Part PR of the Tangent intercepted by the Point of Contact and the Perpendicular CP, is every where equal to the Radius CO of the generating Circle: Therefore $\dot{z}$ $\left(= \frac{y\dot{y}}{t}\right)$* being here $=$ $\frac{y\dot{y}}{a}$, we firſt get $z = \frac{y^2}{2a}$; which corrected, by making $y = a$ ($=$ AC) becomes $\frac{y^2 - a^2}{2a}$ $\left(\frac{CP^2}{2CA}\right)$ the true Meaſure of the required Arch AR.



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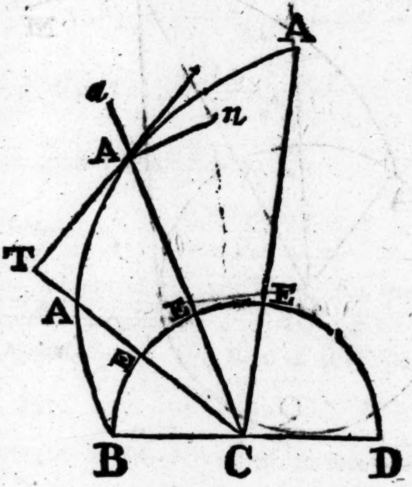
* Art. 136.

EXAMPLE IV.

140. *In which the Spiral of* Archimedes *is propoſed.*

Where, the Value of t (AT) being denoted by $\frac{by}{\sqrt{b^2+y^2}}$ (*Vid. Art.* 62.) we get $\dot{z}$ $\left(=\frac{y\dot{y}}{t}\right)$ $=\frac{\dot{y}\sqrt{b^2+y^2}}{b}$: Which Fluxion being exactly the ſame as that expreſſing the Arch of the common Parabola, found in *Article* 128. its Fluent will therefore be truly repreſented by the Meaſure of the ſaid Arch, or by $\frac{\frac{1}{2}y\sqrt{b^2+y^2}}{b} + \frac{1}{2}b \times$ *hyp. Log.* $\frac{y+\sqrt{b^2+y^2}}{b}$, the Value there exhibited.



EXAMPLE V.

141. *Let the Curve be a Spiral whoſe Equation is* $a^{m-1}x = y^m$ (*Vid. Art.* 136.)

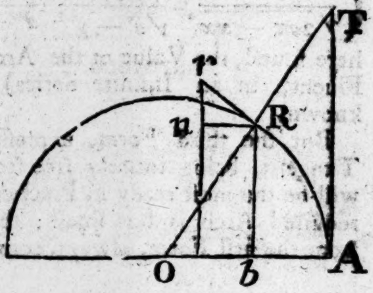
In which Caſe $\dot{x}$ being $= \frac{m\dot{y}y^{m-1}}{a^{m-1}}$, it is evident that $\dot{z}\ (= \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}\,^{*}) = \sqrt{\dot{y}^2 + \frac{m^2y^{2m}\dot{y}^2}{a^{2m}}} = \dot{y}\sqrt{1 + \frac{m^2y^{2m}}{a^{2m}}}$; and therefore $z = y + \frac{m^2y^{2m+1}}{\overline{2m+1}\times 2a^{2m}} - \frac{m^4y^{4m+1}}{\overline{4m+1}\times 8a^{4m}} + \frac{m^6y^{6m+1}}{\overline{6m+1}\times 16a^{6m}}$ &c. Which Value may be otherwiſe had, without an Infinite Series, when $\frac{1}{2m}$ is a whole poſitive Number, *Vid. Art.* 138.

* Art. 136.

EXAMPLE VI.

142. *Where, the Right-ſine, Verſed-ſine, Tangent, or Secant of an Arch of a Circle, being given, it is required to find the Length of the Arch itſelf in Terms thereof.*

Put the Verſed-ſine $Ab = x$, the Right-ſine $Rb = y$, the Tangent AT $= t$, the Secant OT $= s$, the Arch AR $= z$, and the Radius AO, or RO, $= a$; alſo let $Rn = \dot{x}$, $nr = \dot{y}$ and $Rr = \dot{z}$: Since the Angle rnR ($=$ Right-angle) $= ObR$, and rRn ($=$ Right-angle $- nRO$) $= ORb$, the Triangles rRn and ORb are



are therefore equi-angular; and it will be, R*b* (y) : OR (a) :: R*n* $(\dot{x})$: R*r* $(\dot{z}) = \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ (becauſe, by the Property of the Circle $\sqrt{2ax - xx} = y$.) Alſo, O*b* $(\sqrt{a^2 - y^2})$: OR (a) :: *nr* $(\dot{y})$ R*r* $(\dot{z}) = \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$. Theſe two Values exhibit the Fluxion of the Arch in Terms of the Verſed-ſine and Right-ſine reſpectively: But, to get the ſame, in Terms of the Tangent and Secant, we have *(by ſim. Triangles)* OT $(= s = \sqrt{a^2 + t^2})$: OA (a) :: OR (a) : O*b* $= \frac{a^2}{s} = \frac{a^2}{\sqrt{a^2 + t^2}}$: Hence A*b* $= a - \frac{a^2}{s} = a - \frac{a^2}{\sqrt{a^2 + t^2}}$; whoſe Fluxion is therefore $= \frac{a^2\dot{s}}{s^2} = \frac{a^2\dot{t}t}{\overline{a^2 + t^2}|^{\frac{3}{2}}}$: Whence (again by ſimilar Triangles) AT $(= \sqrt{s^2 - a^2} = t)$: OT $(= s = \sqrt{a^2 + t^2})$ :: R*n* : R*r* $= \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}} = \frac{a^2\dot{t}}{a^2 + t^2} = \dot{z}$.

Now, from any one of the four Forms of Fluxions $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}, \frac{a\dot{y}}{\sqrt{a^2 - y^2}}, \frac{a^2\dot{t}}{a^2 + t^2}, \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}\right)$ here found, the Value of the Arch itſelf (by taking the Fluent, in an Infinite Series) will likewiſe become known.

But the third Form, expreſſed in Terms of the Tangent, being intirely free from radical Quantities, will be the moſt ready in Practice, eſpecially where the required Arch is but ſmall; though the Series ariſing from the firſt Form, always, converges the faſteſt.

If, therefore, $\frac{a^2\dot{t}}{a^2+t^2}$ be now converted to an Infinite Series, we ſhall have $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$ *&c.* and conſequently $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ *&c.* = AR. Where, if (*for Example Sake*) AR be ſuppoſed an Arch of 30 Degrees, and AO (to render the Operation more eaſy) be put = Unity, we ſhall have $t = \sqrt{\frac{1}{3}} = .5773502$ (becauſe Ob $\sqrt{\frac{3}{4}}$: bR $(\frac{1}{2})$ :: OA (1) : AT $(t) = \sqrt{\frac{1}{3}}$).

Whence

$$t^3 \;\left(= t \times t^2 = t \times \tfrac{1}{3}\right) = .1924500$$

$$t^5 \;\left(= t^3 \times t^2 = \frac{t^3}{3}\right) = .0641500$$

$$t^7 \;\left(= t^5 \times t^2 = \frac{t^5}{3}\right) = .0213833$$

$$t^9 \;\left(= t^7 \times t^2 = \frac{t^7}{3}\right) = .0071277$$

$$t^{11} \;\left(= t^9 \times t^2 = \frac{t^9}{3}\right) = .0023759$$

$$t^{13} \;\left(= t^{11} \times t^2 = \frac{t^{11}}{3}\right) = .0007919$$

$$t^{15} \;\left(= t^{13} \times t^2 = \frac{t^{13}}{3}\right) = .0002639$$

&c.

And therefore $AR = .5773502 - \frac{.1924500}{3} + \frac{.0641500}{5} - \frac{.0213833}{7} + \frac{.0071277}{9} - \frac{.0023759}{11} +$

$$+\frac{.0007919}{13}-\frac{.0002639}{15}+\frac{.0000879}{17}-\frac{.0000293}{19}$$

$$+\frac{.0000097}{21}-\frac{.0000032}{23}=.5235987$$: Which multiplied by 6 gives 3,141592 + for the Length of the Semi-periphery of the Circle whoſe Radius is Unity.

At *Article* 126. certain Forms of Fluxions were pointed out, whoſe Fluents are explicable by means of hyperbolical Spaces, or a *Table of Logarithms*: Which Forms, it is obſervable, agree in every thing, but the Signs (and conſtant Quantities) with thoſe exhibited above, for the Arch of a Circle. And theſe laſt, like them, may ſerve as ſo many (other) Theorems for finding Fluents by means of a *Table of Sines, Tangents and Secants.* But, as ſuch a Table is uſually calculated to a Radius of 1,000000 *&c.* (or Unity) the following Equations, derived from thoſe above, being adapted to that Radius, will be rather more commodious.

$$\text{Thus, the Fluent of}\left\{\begin{matrix}\dfrac{\dot{w}}{\sqrt{2aw-w^2}}\\[2ex] \dfrac{\dot{w}}{\sqrt{a^2-w^2}}\\[2ex] \dfrac{a\dot{w}}{a^2+w^2}\\[2ex] \dfrac{a\dot{w}}{w\sqrt{w^2-a^2}}\end{matrix}\right\}\text{is equal to the Arch whoſe}\left\{\begin{matrix}\text{Verſed-ſine}\\[2ex] \text{Right-ſine}\\[2ex] \text{Tangent}\\[2ex] \text{Secant}\end{matrix}\right\}\text{is }\frac{w}{a}\text{, and Radius Unity.}$$

The way of deducing theſe Expreſſions, from the foregoing ones, is extremely eaſy: For, if A be put to denote the Arch whoſe Radius is Unity, and whoſe Verſed-ſine, Right-ſine, Tangent, or Secant is $\frac{w}{a}$ (according to the different Caſes here ſpecified). Then, becauſe ſimilar Arcs, of unequal Circles, are as their

Radii, it will be $1 : a :: A : (aA)$ the Length of the Arch AR (*ſee the Figure.*) Therefore, the Fluent of $\frac{a\dot{x}}{\sqrt{2ax - xx}}$ (or $\frac{a\dot{w}}{\sqrt{2aw - w^2}}$, putting $w = x$) being $= aA$ (AR), that of $\frac{\dot{w}}{\sqrt{2aw - w^2}}$ muſt neceſſarily be $= A$: And in the very ſame Manner the other Forms are made out.

EXAMPLE VII.

143. *Let the propoſed Curve be the common Cycloid.*

Then, if the Radius AO of the generating Semi-circle* be denoted by a, we ſhall have $BR = \sqrt{2ax - x^2}$; and the Fluxion thereof $= \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}}$: Which being added to $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}\right)$ the Fluxion of AR or its Equal RS (given by the preceding Article) we thence get $\frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}} = \frac{\dot{x} \times \overline{2a - x}}{x^{\frac{1}{2}} \times \overline{2a - x}|^{\frac{1}{2}}} = \frac{\dot{x}}{x^{\frac{1}{2}}} \times \overline{2a - x}|^{\frac{1}{2}}$, for the true Fluxion of the Ordinate BS of the Cycloid.

Hence $\dot{z}$ $(\sqrt{\dot{x}^2 + \dot{y}^2}$ †$) = \sqrt{\dot{x}^2 + \frac{\dot{x}^2 \times \overline{2a - x}}{x}} = \dot{x}\sqrt{\frac{2a}{x}} = \overline{2a}|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$; and conſequently, by taking the Fluent, $z = \overline{2a}|^{\frac{1}{2}} \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{2ax} =$ the Arch AS of the Cycloid.

EX-

* See Fig. Art. 132.

† Art. 135.

EXAMPLE VIII.

144. *Wherein it is required to determine the Length of the Arch of the common Hyperbola.*

In this Caſe (the Semi-tranſverſe Axis being repreſented by b, and the Semi-conjugate by c) we have $\frac{b^2y^2}{c^2} = 2bx + x^2$; and therefore $x = \frac{b\sqrt{c^2+y^2}}{c} - b$: Hence $\dot{x} = \frac{by\dot{y}}{c\sqrt{c^2+y^2}}$, and $\dot{z}$ $(= \sqrt{\dot{y}^2+\dot{x}^2})$ $\sqrt{\dot{y}^2 + \frac{b^2y^2\dot{y}^2}{c^2 \times \overline{c^2+y^2}}} = \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4+c^2y^2}}$; which, by converting $\frac{b^2y^2}{c^4+c^2y^2}$ into an *Infinite Series*, becomes $\dot{y}\sqrt{1 + \frac{b^2y^2}{c^4} - \frac{b^2y^4}{c^6} + \frac{b^2y^6}{c^8} - \frac{b^2y^8}{c^{10}} \text{ \&c.}}$ But ſtill we have the Square Root to extract; In order thereto let it be aſſumed $= 1 + Ay^2 + By^4 + Cy^6 + Dy^8$ &c. Then, by ſquaring, and tranſpoſing (*Vid. Art.* 98.) there ariſes

$$\left.\begin{array}{l} 1 + 2Ay^2 + 2By^4 + 2Cy^6 + 2Dy^8 \text{ \&c.} \\ \qquad\quad + A^2y^4 + 2ABy^6 + 2ACy^8 \text{ \&c.} \\ \qquad\qquad\qquad\quad + B^2y^8 \text{ \&c.} \\ -1 - \frac{b^2}{c^4} \times y^2 + \frac{b^2}{c^6} \times y^4 - \frac{b^2}{c^8} \times y^6 + \frac{b^2}{c^{10}} \times y^8 \text{ \&c.} \end{array}\right\} = 0$$

Hence $A = \frac{b^2}{2c^4}$; $B = -\frac{b^2}{2c^6} - \frac{1}{2}A^2 = -\frac{b^2}{2c^6} - \frac{b^4}{8c^{0}}$; $C = \frac{b^2}{2c^8} - AB = \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}$; &c. &c. Therefore $\dot{z}$ $(= \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4} \text{ \&c.}} = \dot{y} \times \overline{1 + Ay^2 + By^4 \text{ \&c.}}) = \dot{y} + \frac{b^2}{2c^4} \times y^2\dot{y} - \overline{\frac{b^2}{2c^6} + \frac{b^4}{8c^8}} \times y^4\dot{y}$

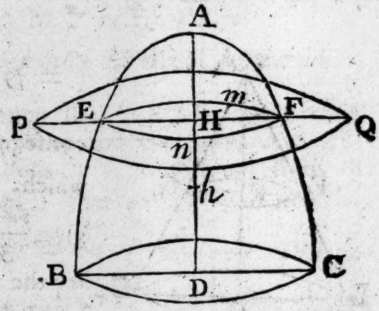
$\overline{y^4\dot{y} + \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}} \times y^6\dot{y}$ &c. And conſequently $z = y + \frac{b^2y^3}{6c^4} - \overline{\frac{b^2}{c^2} + \frac{b^4}{4c^4}} \times \frac{y^5}{10c^4} + \overline{\frac{b^2}{c^2} + \frac{b^4}{2c^4} + \frac{b^6}{8c^6}} \times \frac{y^7}{14c^6}$ &c.

By the very ſame way of proceeding the Arch of an Ellipſis may be found, the Equations of the two Curves differing in nothing but their Signs.

SECTION IX.

The Application of FLUXIONS *in inveſtigating the Contents of Solids.*

145. LET ABC repreſent any Solid; conceived to be generated (or deſcribed) by a Plane PQ paſſing over it, with a parallel Motion: Let H*h* (perpendicular to PQ) be taken to expreſs the Fluxion of AH (x) or the Velocity with which the generating Plane is carry'd; alſo let the Area of the Part, E*m*F*n*, of the Plane intercepted by, or contained in, the Solid, be denoted by A: Then it follows, from *Art.* 2 and 5. that the Fluxion of the Solid AEF, will be expreſſed by $A\dot{x}$. From whence, by expounding A in Terms of x, (according to the Nature of the Figure) and then taking the Fluent, the Content of



of the Solid (which we ſhall, always, hereafter repreſent by s) will be given.

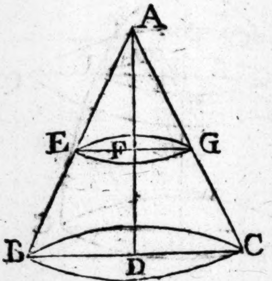
But, when the propoſed Solid is that ariſing from the Revolution of any given Curve AEB about AHD, as an Axis, the Fluxion ($\dot{s}$) of the Solidity may be exhibited in a Manner more convenient for Practice: For, putting the Area (3,141592 &c.*) of the Circle, whoſe Radius is Unity, $= p$, and the Ordinate EH $= y$, it will be $1^2 : y^2 :: p : (py^2)$ the Area of the Circle EmFn, which being wrote above inſtead of A, we have $\dot{s} = py^2\dot{x}$. The Uſe of which will be ſufficiently ſhewn in the following Examples.

* Art. 124.

EXAMPLE I.

146. *Let it be propoſed to find the Content of a Cone* ABC.

Put the given Altitude (AD) of the Cone $= a$, and the Semi-diameter (BD) of its Baſe $= b$: Then, the Diſtance (AF) of the Circle EG, from the Vertex A, being denoted by x, &c. we have, by ſimilar Triangles, as $a : b :: x :$ EF $(y) = \frac{bx}{a}$. Whence, in this Caſe, $\dot{s}$



$(= py^2\dot{x}) = \frac{pb^2x^2\dot{x}}{a^2}$; and conſequently $s = \frac{pb^2x^3}{3a^2}$; which, when $x = a$ (= AD) gives $\frac{pb^2a}{3}$ ($= p \times \text{BD}^2 \times \frac{1}{3}\text{AD}$) for the Content of the whole Cone ABC. Which appears, from hence, to be juſt $\frac{1}{3}$ of a Cylinder of the ſame Baſe and Altitude.

EXAMPLE II.

147. *Where, let the Solid propoſed be a parabolic Conoid, or that ariſing from the Revolution of any Kind of Parabola about its Axis.*

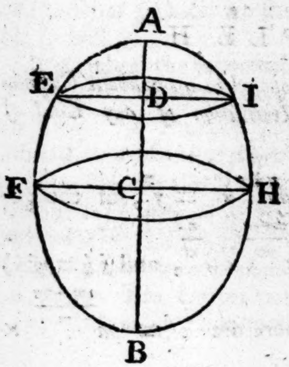
Then, from the Equation $a^{m-n}x^{n}=y^{m}$, of the generating Curve, we get $y=a^{\frac{m-n}{m}}\times x^{\frac{n}{m}}$, and $\dot{s}\,(=py^2\dot{x})$ $=pa^{\frac{2m-2n}{m}}\times\dot{x}x^{\frac{2n}{m}}$; and therefore $s=pa^{\frac{2m-2n}{m}}\times\dfrac{x^{\frac{2n}{m}+1}}{\frac{2n}{m}+1}=pa^{\frac{2m-2n}{m}}\times\dfrac{mx^{\frac{2n}{m}+1}}{2n+m}=pa^{\frac{2m-2n}{m}}\times x^{\frac{2n}{m}}\times$ $\dfrac{mx}{2n+m}=py^2\times\dfrac{mx}{2n+m}=$ the Content of the Solid; which therefore is to (py^2x) the Content of the circumſcribing Cylinder, as m to $2n+m$. Whence the Solid generated by the conical Parabola (where $m=2$, and $n=1$) appears to be juſt $\frac{1}{2}$ of its circumſcribing Cylinder.

EXAMPLE III.

148. *Let the propoſed Solid* AFBH *be a Spheroid.*

In which Caſe, putting the Axis AB, about which the Solid is generated, $=a$, and the other Axis FH, of the generating Ellipſis $=b$, it follows, from the Property of the Ellipſis, that $a^2:b^2::x\times\overline{a-x}$ (AD × BD) : y^2 $(\text{DE})^2=\dfrac{b^2}{a^2}\times\overline{ax-xx}$: Whence we have $\dot{s}\,(=py^2\dot{x}^{*})=\dfrac{pb^2}{a^2}\times\overline{a x\dot{x}-x^2\dot{x}}$; and $s=\dfrac{pb^2}{a^2}\times\overline{\tfrac{1}{2}axx-\tfrac{1}{3}x^3}=$ the Segment AIE. Which, when

* Art. 145.



when AD (x) = AB (a), becomes $\left(\frac{pb^2}{a^2}\times\overline{\tfrac{1}{2}a^3-\tfrac{1}{3}a^3}\right)$ $\frac{1}{6}pab^2$ = the Content of the whole Spheroid. Where, if b (FH) be taken = a (AB) we ſhall alſo get $\frac{1}{6}pa^3$ for the true Content of the Sphere whoſe Diameter is a. Hence a Sphere, or a Spheroid, is $\frac{2}{3}$ of its circumſcribing Cylinder; for the Area of the Circle FH being expreſſed by $\frac{pb^2}{4}$, the Content of the Cylinder whoſe Diameter is FH, and Altitude AB, will therefore be $\frac{pb^2a}{4}$; of which $\frac{1}{6}pab^2$, is, evidently, two third Parts.

EXAMPLE IV.

149. *Let the Solid, whoſe Content you would find, be the hyperbolical Conoid.*

Then, from the Equation, $y^2=\frac{b^2}{a^2}\times\overline{ax+xx}$, of the generating Hyperbola, we have $\dot{s}$ $(py^2\dot{x})=\frac{pb^2}{a^2}\times\overline{ax\dot{x}+x^2\dot{x}}$, and conſequently $s=\frac{pb^2}{a^2}\times\overline{\tfrac{1}{2}ax^2+\tfrac{1}{3}x^3}$ = the Content of the Conoid; which therefore is to $\left(\frac{pb^2}{a^2}\times\overline{ax+x^2}\times x\right)$ that of a Cylinder of the ſame Baſe and Altitude, as $\frac{1}{2}a+\frac{1}{3}x$ to $a+x$. This Ratio, if x be extremely ſmall, will become as 1 to 2 very nearly: Whence it may be inferr'd, that the Content of a very ſmall Part of any Solid, generated by a Curve, whoſe Ray of Curvature at the Vertex is a finite Quantity, is half that of a Cylinder of the ſame Baſe and Altitude, very nearly: Becauſe any ſuch Curve, for a ſmall Diſtance, will differ inſenſibly from an Hyperbola, whoſe Radius of Curvature, at the Vertex, is the ſame.

This might have been inferred, either, from the common parabolic Conoid, or the Spheroid, in the preceding Examples; but other Obſervations would not allow Room for it there.

EXAMPLE V.

150. *In which the propoſed Solid is that ariſing from the Rotation of the Ciſſoid of* Diocles, *about its Axis.*

Here, y^2 being $= \frac{x^3}{a - x}$, * we have $\dot{s}\ (py^2\dot{x}) \doteq$ $\frac{px^3\dot{x}}{a-x}$. But, in Caſes like this, (where the Denominator is rational and the variable Quantity in the Numerator of ſeveral Dimenſions) it will be neceſſary to divide the latter by the former, in order to obtain the Fluent, by leſſening the Number of Dimenſions: Thus, dividing $px^3\dot{x}$ by $-x+a$, according to the Manner of compound Quantities, the Work will ſtand thus:

$$\begin{array}{lllll}
-x+a) & px^3\dot{x}-0 & & & (-px^2\dot{x}-pax\dot{x}-pa^2\dot{x} \\
& \underline{px^3\dot{x}-pax^2\dot{x}} & & & \\
& & +pax^2\dot{x}-0 & & \\
& & \underline{+pax^2\dot{x}-pa^2x\dot{x}} & & \\
& & & +pa^2x\dot{x}-0 & \\
& & & \underline{+pa^2x\dot{x}-pa^3\dot{x}} & \\
& & & \underline{\qquad\quad +pa^3\dot{x}}\,. &
\end{array}$$

Where, the Quotient being $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}$, and the Remainder $pa^3\dot{x}$, the Value of the given Fraction $\frac{px^3\dot{x}}{a-x}$, will

* Art. 56.

will therefore be truly expreſſed by $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}+\frac{pa^3\dot{x}}{a-x}$: Whoſe Fluent, properly corrected, is $-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times hyp.\ Log.\ \frac{a}{a-x}$ *Vid. Art.* 126.

EXAMPLE VI.

151. *Let the Solid be that ariſing from the Rotation of the Conchoid of* Nicomedes *about its Axis.*

The Sub-tangent $\frac{y\dot{x}}{\dot{y}}$ of this Curve being $=\frac{-ab^2-y^3}{y\sqrt{b^2-y^2}}$ (*Vid. Art.* 48 *and* 57.) we have $\dot{x}=\frac{-ab^2\dot{y}-y^3\dot{y}}{y^2\sqrt{b^2-y^2}}$, and

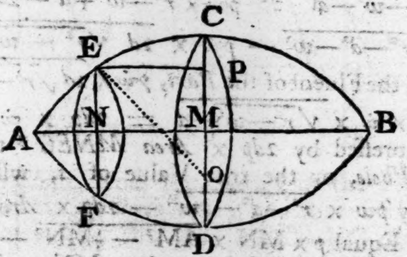
* Art. 145. therefore $\dot{s}\ (py^2\dot{x}\,*)=\frac{-pab^2\dot{y}-py^3\dot{y}}{\sqrt{b^2-y^2}}=-\frac{pab^2\dot{y}}{\sqrt{b^2-y^2}}-\frac{py^3\dot{y}}{\sqrt{b^2-y^2}}$. But, in order for the more eaſy finding the Fluent thereof, put $\sqrt{b^2-y^2}=u$; and then, y being $=\sqrt{b^2-u^2}$, and $\dot{y}=\frac{-u\dot{u}}{\sqrt{b^2-u^2}}$, we ſhall, by Subſtitution, get $\dot{s}=\frac{pab^2\dot{u}}{\sqrt{b^2-u^2}}+p\times\overline{b^2\dot{u}-u^2\dot{u}}$. Whence, the Fluent of $\frac{\dot{u}}{\sqrt{b^2-u^2}}$ being expreſſed by the Arch (A) of the Circle whoſe Radius is Unity and

† Art. 142. Sine $\frac{u}{b}$ †, the Fluent of the whole Expreſſion will be $pab^2\times A+p\times\overline{b^2u-\frac{1}{3}u^3}$. Which, when $y=0$, or $u=b$, gives $(pab^2\times\frac{1}{2}p+p\times\frac{2}{3}b^3)\ pb^2\times\overline{\frac{1}{2}pa+\frac{2}{3}b}$ for the Content of the whole Solid, when its Axis becomes infinite.

EXAMPLE VII.

152. *Where it is required to find the Content of a parabolic Spindle; generated by the Rotation of a given Parabola* ACB *about its Ordinate* AB.

Put CM (the Abſciſſa of the given Parabola) $= a$, and the Semi-ordinate AM (or BM) $= b$; and, ſuppoſing ENF to be any Section of the Solid parallel to DC, let its Diſtance MN (or EP) from DC, be denoted by w: Then, by the Property of the Curve, we ſhall



have AM² (b^2) : EP² (w^2) :: CM (a) : CP $= \frac{aw^2}{b^2}$: Therefore EN $(=$ CM $-$ CP$)$ $= a - \frac{aw^2}{b^2} = \frac{a \times \overline{b^2 - w^2}}{b^2}$, and conſequently $p \times$ EN² $= \frac{pa^2}{b^4} \times \overline{b^4 - 2b^2w^2 + w^4} =$ the Area of the Section EF: Which multiply'd by $(\dot{w})$ the Fluxion of MN, gives $\frac{pa^2}{b^4} \times \overline{b^4\dot{w} - 2b^2w^2\dot{w} + w^4\dot{w}}$ for the Fluxion of the Solidity, * whoſe Fluent, $\frac{pa^2}{b^4} \times \overline{b^4w - \frac{2}{3}b^2w^3 + \frac{1}{5}w^5}$, when w becomes $= b$, is $\left(\frac{8pa^2b}{15}\right)$ half the Content of the Solid.

* Art. 145.

EXAMPLE VIII.

153. *Let the Solid* ACBD *(ſee the laſt Figure) be a Spindle, generated by the Rotation of the Segment of a Circle,* ACB, *about its Chord, or Ordinate,* AB.

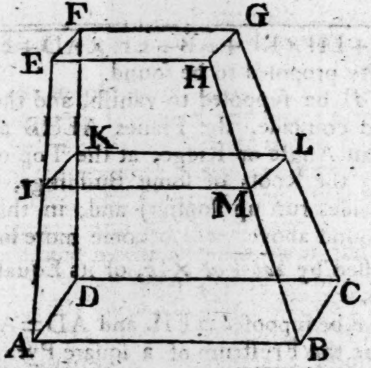
Then, if the Radius OE be put $= r$, OM $= d$, and EP $= w$ *&c.* (as before) we ſhall have OP ($= \sqrt{\overline{OE^2 - EP^2}}$) $= \sqrt{\overline{r^2 - w^2}}$, and EN ($=$ OP$-$OM) $= \sqrt{\overline{r^2 - w^2}} - d$: Therefore $\dot{s}$, in this Caſe, is $= p\dot{w} \times \overline{\sqrt{\overline{r^2 - w^2}} - d}\,|^2 = p\dot{w} \times \overline{r^2 - w^2 + d^2 - 2d\sqrt{r^2 - w^2}}$ $= p\dot{w} \times \overline{r^2 - d^2 - w^2} - p\dot{w} \times \overline{2d\,\sqrt{\overline{r^2 - w^2}} - 2d^2}$: Whence, the Fluent of the *Part*, $p\dot{w} \times 2d\sqrt{\overline{r^2 - w^2}} - 2d^2$ ($= 2dp \times \dot{w} \times \overline{\sqrt{\overline{r^2 - w^2}} - d} = 2dp \times \dot{w} \times$ EN) being expreſſed by $2dp \times$ *Area MNEC* * the Fluent of the *Whole*, or the true Value of s, will be expreſſed by $pw \times \overline{r^2 - d^2 - \frac{1}{3}w^2} - 2dp \times$ *Area MNEC*, or by its Equal $p \times \text{MN} \times \overline{\text{AM}^2 - \frac{1}{3}\text{MN}^2} - 2p \times \text{OM}$ $\times$ *Area MNEC*: Which, when MN $=$ MA, gives $p \times \frac{2}{3}\text{AM}^3 - 2p \times \text{OM} \times$ *Area ACM*, for the Content of half the Solid: Where the *Area ACM* may be found by *Art.* 124. or more eaſily by the common Table of the Areas of the Segments of a Circle; to be met with in moſt Books of *Gauging*.

* Art. 112.

EXAMPLE IX.

154. *Let it be propoſed to find the Content of the Solid* AEGB; *whoſe four Sides* AH, AF, CH, CF *are plane Surfaces, and its Ends* ADCB, EFGH *given Rectangles, parallel to each other.*

Let the Sides AB and AD, of the Baſe, be denoted by a and b; and thoſe of the Top (EH and EF) by c and d reſpectively; moreover, let h expreſs the perpendicular Height of the Solid; and let x (conſider'd as variable) be the Diſtance of (IL) any Section thereof (parallel to the Baſe) from the Plane EG.



It is evident, from the Nature of the Figure, that the Section IL is a Rectangle; and that $b : x :: AB-EH : IM-EH :: BC-HG : ML-HG$.

From theſe Proportions we have $IM-EH = \frac{\overline{a-c}\times x}{b}$ and $ML-HG = \frac{\overline{b-d}\times x}{b}$: Hence $IM = \frac{\overline{a-c}\times x}{b} + c$, and $ML = \frac{\overline{b-d}\times x}{b} + d$; and conſequently the Area of the Rectangle (IL) $= \frac{\overline{a-c}\times\overline{b-d}}{b^2}\times x^2 + \frac{ad-2cd+cb}{b}\times x + cd$: Which being multiply'd by $\dot{x}$, and the Fluent taken, there reſults $\frac{\overline{a-c}\times\overline{b-d}\times x^3}{3b^2} + \frac{\overline{ad-2cd+cb}\times x^2}{2b} + cdx$ for the Content of IFGL:

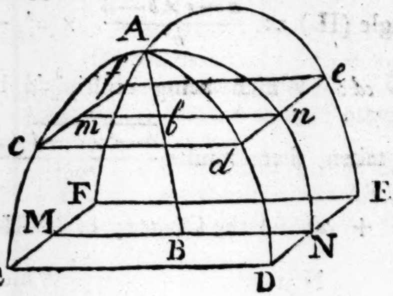
Which, when $x=h$, becomes $\left(\frac{\overline{a-c}\times\overline{b-d}\times h}{3}+\frac{\overline{ad-2cd+cb}\times h}{2}+cdh=\overline{2ab+ad+bc+2cd}\times\frac{1}{6}h=\right)$ $\overline{AB\times AD+EH\times EF+\overline{AB+EH}\times\overline{AD+EF}}\times\frac{1}{6}h=$ the Quantity propoſed to be found.

If EF (d) be ſuppoſed to vaniſh, and the Lines EH and FG to coincide, the Planes AEHB and DFGC will form an Angle or Ridge, at the Top of the Solid (reſembling the Roofs of ſome Buildings, whoſe Ends as well as Sides run up ſloping) and, in this Caſe, the Content, found above, will become more ſimple, being then expreſſed by $\overline{2ab+bc}\times\frac{1}{6}h$, or its Equal $\overline{2AB+EH}\times AD\times\frac{1}{6}h$.

But, if EF be ſuppoſed $=$ EH, and AD $=$ AB, the Solid will then be the Fruſtrum of a ſquare Pyramid; and its Content $=\overline{a^2+ac+c^2}\times\frac{1}{3}h$, $=\overline{AB^2+AB\times EH+EH^2}\times\frac{1}{3}h$: From whence, by taking EH $=0$, the Content of the whole Pyramid whoſe Baſe is AB^2, and its Altitude h, will alſo be given, being $=AB^2\times\frac{1}{3}h$.

EXAMPLE X.

155. *Let the propoſed Solid be that, commonly known by the Name of a Groin*; whoſe Sections parallel to the Baſe are, all, Squares, and whereof the two Sections perpendicular to the Baſe, through the Middle of the oppoſite Sides, are Semi-circles.



Let $bcdef$ be any Section parallel to the Baſe; and let its Diſtance Ab from the Vertex of the Solid, be denoted by x; alſo let a repreſent the Radius AB (or BN) of the circular Section ABNA, perpendicular to the Baſe. Then, *bn* being (by the Property of the Circle) $= \sqrt{2ax-xx}$, the Side of the Square *df*, will be $= 2\sqrt{2ax-xx}$, and therefore the Area $= 4 \times \overline{2ax-xx}$; whence $\dot{s} = 4\dot{x} \times \overline{2ax-xx}$, and conſequently $s = 4ax^2 - \frac{4x^3}{3}$: Which, when $x = a$, becomes $\frac{\overline{2a}|^3}{3} =$ the Content of the whole Solid.

If the Solid be a Groin of any other Kind, or ſuch, that its two Sections perpendicular to the Baſe, through the Middle of the oppoſite Sides, are any other Curves than Semi-circles, the Content may, ſtill, be found in the ſame Manner; and will be always in proportion to the Solid generated by the Revolution of the ſaid Curve about its Axis, as a Square, is to its inſcribed Circle. But, if the foreſaid perpendicular Sections be Curves of different Kinds, the Sections parallel to the Baſe will no longer be Squares, but Rectangles; whoſe Sides are the correſponding (double) Ordinates of the reſpective Curves. Thus, for Inſtance, let one Section be a Circle and the other a Parabola, whoſe Ordinates, to the common Abſciſſa, x, are expreſſed by $\sqrt{dx-xx}$ and $\sqrt{ax}$, reſpectively; then the Sides of the rectangular Section, parallel to the Baſe of the Groin, will be $2\sqrt{dx-xx}$ and $2\sqrt{ax}$: Whence the Area of that Section is $= 4x\sqrt{ad-ax}$, and therefore $\dot{s} = 4x\dot{x}\sqrt{ad-ax}$: Where, by taking the Fluent, * $s =$ $\frac{16d^2\sqrt{ad} - a^{\frac{1}{2}} \times \overline{d-x}|^{\frac{3}{2}} \times \overline{16d+24x}}{15} =$ the true Content of ſuch a Solid.

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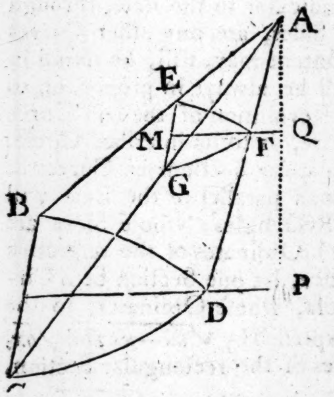
* Art. 83.

EXAMPLE XI.

156. *Where the Solid* BACD *propoſed is a kind of Cone, or Pyramid*; form'd by conceiving Right-lines to be drawn from every Point in the Perimeter of any given Plane BDC, to a given Point, or Vertex, A above that Plane.

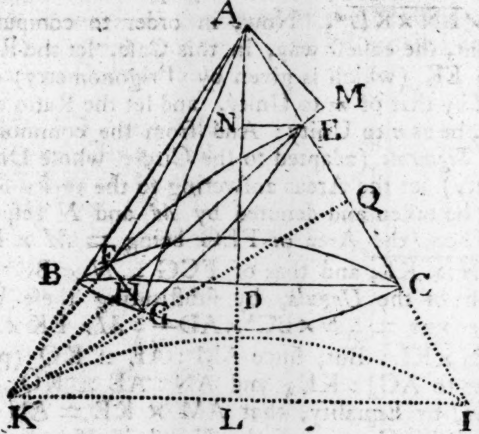


Let EFG be any Section parallel to BDC, whoſe perpendicular Diſtance (AQ) from the Vertex let be denoted by x; moreover, let the whole given Altitude (AP) of the Solid be put $= a$, and the Area of the Baſe BDC (which is alſo ſuppoſed given) $= b$.

In the firſt place, it is eaſy to conceive that the Planes BDC and EFG muſt be ſimilar: And therefore, ſince ſimilar Figures are to each other as the Squares of their like Sides, or Dimenſions, it follows that $AP^2\ (a^2) : AQ^2\ (x^2) :: BDC\ (b) : EFG = \frac{bx^2}{a^2}$. Whence $\dot{z} = \frac{bx^2\dot{x}}{a^2}$, and conſequently $z = \frac{bx^3}{3a^2} = \frac{ba}{3}$, when $x = a$. Therefore the Solidity of a Cone or Pyramid, let the Figure of its Baſe be what it will, is always had by multiplying the Area of the Baſe by $\frac{1}{3}$ of the Altitude.

EXAMPLE XII.

157. *Where it is propoſed to find the Content of the Ungula* EFGC, cut off from a given Cone, ABC, by a Plane EFG paſſing through the Baſe thereof.



Let AD be the perpendicular Height of the Cone, alſo let AM be perpendicular to HE, the Axis of the Section FEG, and let FAG be another Section of the Cone, thro' FG and the Vertex A.

Since the Solids CAFG and EAFG, whoſe Baſes are FCG, and FEG, come under the Form ſpecified in the preceding Example, their Contents will therefore be expreſſed by $FCG \times \frac{1}{3} AD$ and $FEG \times \frac{1}{3} AM$ reſpectively: Whoſe Difference, $\frac{FCG \times AD - FEG \times AM}{3}$, is the Solidity of the *Ungula* CEFG: Where the Baſes FCG and FEG being conic Sections, their Areas will be given by *Art.* 115. 124 and 129. from whence the whole will be known. Thus, if HE be ſuppoſed parallel to AB, the Section FEG, then being a Parabola, its Area will be $= \frac{2}{3} \times FG \times EH$ *: Whence the Solidity of the

*Art. 115.

Segment EFGA is $= \frac{2}{9} \times FG \times EH \times AM$: Which being deducted from that of CFGA (found by Help of the common Table of circular Segments) the Remainder will be the Content of the *Ungula*. But, if the Axis EH produced, cuts AB, the Section FEG will be a Segment of an Ellipſis EFKG; whoſe conjugate Axis (ſuppoſing EN and KL perpendicular to AD) is $= 2\sqrt{EN \times KL}$ *. Now, in order to compute the Content, the eaſieſt way, in this Caſe, let the Ratio of EH to EK (which is given by Trigonometry) be expreſſed by that of *m* to Unity, and let the Ratio of CH to CB, be as *n* to Unity: And from the common Table of *Segments* (adapted to the Circle whoſe Diameter is Unity) let the Areas anſwering to the verſed Sines *m* and *n*, be taken and denoted by *M* and *N* reſpectively: Then, the Area of FEG being $= M \times EK \times 2\sqrt{EN \times KL}$, and that of FCG $= N \times BC^2$ †, the Content of the *Ungula*, by ſubſtituting theſe Values, will become $= \frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times EK \times AM \times 2\sqrt{EN \times KL}$: But, ſince AM : AE :: KQ (perpendicular to AC) : KE; and AN : AE :: KQ : KI, it follows, by Equality, that $AM \times KE = AN \times KI$; whence the Content of the *Ungula* is alſo expreſſed by $\frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times AN \times KI \times 2\sqrt{EN \times KL}$. Which, if H be ſuppoſed to coincide with B, and KI with BC, will become $\frac{(0.78539}{3}$ &c. $\times BC^2 \times AD - \frac{0.78539}{3}$ &c. $\times AN \times BC \times 2\sqrt{EN \times BD} = 0.26179$ &c. $\times BC \times \overline{BC \times AD - 2AN \times \sqrt{EN \times BD}}$.

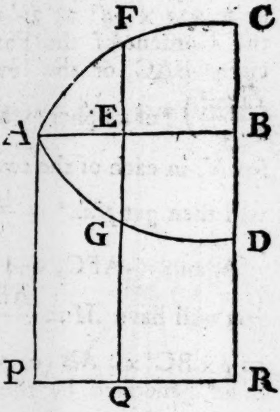
When the Section EFG is an Hyperbola, its Area may be found by means of a Table of Logarithms (inſtead of a Table of Segments) whence the Content of the *Ungula* will likewiſe be had in that Caſe.

* Art. 41.

† Art. 124 and 130.

EXAMPLE XIII.

158. *Let* AFC, *or* AGD, *be a Curve of any Kind*; whoſe Area, and the Content of the Solid ariſing from its Rotation about its Axis, or Ordinate, AB, are both known; it is propoſed to find, from thence, the Content of the Solid generated by the Revolution of that Curve about any other Line PR parallel to the ſaid Axis or Ordinate AB.



Let AP, FQ, and CR be all perpendicular to AB and to the Axis of Motion PQR; alſo let AP (or EQ) = *a*, AE, conſidered as variable, = *w*, the Area AFE, or AEG = *M*, and the Solid, ariſing from its Revolution about AB, = *N*. It is plain that the Area of the Circle generated by QF will be $= p \times FQ^2$ * $= p \times \overline{a + EF}^2 = pa^2 + 2pa \times EF + p \times EF^2$; from which deducting the Area, pa^2, generated by QE, the Remainder, $2pa \times EF + p \times EF^2$, will be the Area of the Annulus generated by EF: Whence the Fluxion of the Solid generated by AEF is truly repreſented by $2pa \times EF \times \dot{w} + p\dot{w} \times EF^2$ †: And, in the ſame manner, it will appear that the Fluxion of the Solid generated by AEG is $2pa \times EG \times \dot{w} - p\dot{w} \times EG^2$. But the Fluent of $EF \times \dot{w}$ (or $EG \times \dot{w}$) is = the Area (*M*) of AEF (or AEG) ‡, and that of $p\dot{w} \times EF^2$ (or $p\dot{w} \times EG^2$) equal to (*N*) the given Solid ariſing from that Area §; therefore the Fluent of the *Whole*, or the Solidity required, is $2paM + N$, in the former Caſe, and $2paM - N$ in the latter; where $2pa$,

in

* Art. 145.

† Art. 145.

‡ Art. 112.

§ Art. 145.

in either Caſe, expreſſes the Periphery of the Cylinder deſcribed by AB, about the Axis of Rotation PR.

Hence, if ABC and ABD are equal and ſimilar to each other, then the Value of *M* &c. being the ſame in both Caſes, it follows that the Content of the Solid generated by AFG will be expreſſed by $2pa \times 2M$, or $2pa \times$ Area AFG.

Now, if (for Example ſake) ACD be ſuppoſed a Circle, whoſe Semi-diameter is *d*, the Area of that Circle being $= pd^2$, the Solid generated by its Revolution (repreſenting the Ring of an Anchor) will therefore be $= 2pa \times pd^2 = 2p^2ad^2$. But if you would know the Content of the Part generated by the upper Semicircle BAC, or the lower one BAD, let the Content $\left(\frac{4pd^3}{3}\right)$ * of a Sphere whoſe Semi-diameter is *d*, be wrote for *N*, in each of the two foregoing Expreſſions, and you will then get $p^2ad^2 + \frac{4pd^3}{3}$, and $p^2ad^2 - \frac{4pd^3}{3}$.

* Art. 148.

Again, if AFC, and AGD be taken as Right-lines, you will have $M = \frac{AB \times BC}{2}$ (or $\frac{AB \times BD}{2}$) and $N = p \times BC^2 \times \frac{1}{3} AB$ (or $p \times BD^2 \times \frac{1}{3} AB$) †: Hence the Solid generated by the Triangle ABC is $(= 2pa \times \frac{AB \times BC}{2} + \frac{p}{3} \times BC^2 \times AB) = p \times AB \times BC \times \overline{RB + \frac{1}{3} BC}$; and that generated by ABD $(= 2pa \times \frac{AB \times BD}{2} - \frac{p}{3} \times BD^2 \times AB) = p \times AB \times BD \times \overline{RB - \frac{1}{3} BD}$.

† Art. 146.

Laſtly, let ABC (or ABD) be conſidered as a Parabola, whoſe Ordinate is AB, and Axis CB (or DB): Then *M* being here $= \frac{2}{3} AB \times BC$ (or $\frac{2}{3} AB \times BD$) ‡ and $N = \frac{8p}{15} \times AB \times BC^2$ § (or $\frac{8p}{15} \times AB \times BD^2$) it

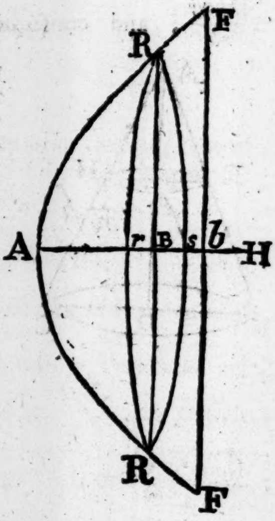
‡ Art. 115.

§ Art. 152.

it follows that the Solid generated by ABC will be ($= 2pa \times \frac{2}{3} AB \times BC + \frac{8p}{15} \times AB \times BC^2$) $= 4p \times AB \times BC \times \frac{5BR + 2BC}{15}$, and that generated by ABD $= 4p \times AB \times BD \times \frac{5BR - 2BD}{15}$.

SECTION X.

The Uſe of Fluxions in finding the Superficies of ſolid Bodies.



159. LET FAF repreſent a Solid generated by the Revolution of any given Curve AF about its Axis AH; alſo let a Circle, whoſe Diameter is the variable Line (or Ordinate) RBR, be conceived to move uniformly from A towards FF, and to dilate itſelf ſo, on all Sides, at the ſame time, as to generate, by its Periphery, the propoſed Superficies RAR: Then the Length of that Periphery, or the generating Line, being expreſſed by 3,141592 * *&c.* × RR ($= 2py$) and the Celerity with which it moves by $\dot{z}$ † the Fluxion of the Superficies RAR, or the Space that

* Art. 142.

† Art. 135.

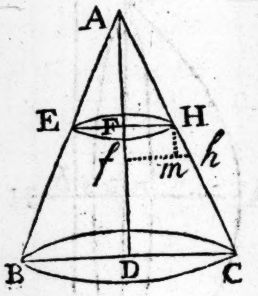
would be uniformly generated in the time of deſcribing $\dot{z}$, will therefore be truly repreſented by $2py\dot{z}$.

Hence, if w be taken to repreſent the whole Surface RAR, generated from the beginning (according to the Method obſerved in the three laſt Sections) we ſhall
* Art. 135. have $\dot{w} = 2py\dot{z} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$ * ; whence w itſelf may be found.

EXAMPLE I.

160. *Let it be propoſed to determine the convex Superficies of a Cone* ABC,

Then, the Semi-diameter of the Baſe (BD, or CD) being put $= b$, the ſlanting Line, or Hypothenuſe, AC $= c$, and FH (parallel to DC) $= y$ &c. we ſhall, from the Similarity of the Triangles ADC and H*mb*,
† Art. 159. have $b : c :: \dot{y}\ (mb) : \dot{z}\ (\mathrm{H}b) = \frac{c\dot{y}}{b}$: Whence $\dot{w}$ ($2py\dot{z}$ †) $= \frac{2pcy\dot{y}}{b}$; and conſequently $w = \frac{pcy^2}{b}$. This, when $y = b$, becomes $= pcb = p \times \mathrm{DC} \times \mathrm{AC} =$ the convex Superficies of the whole Cone ABC: Which therefore is equal to a Rectangle under half the Circumference of the Baſe and the ſlanting Line.

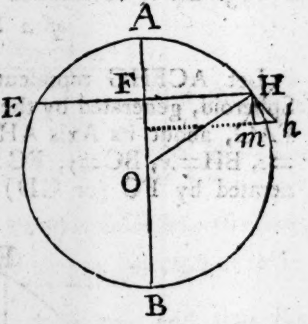


E X,

EXAMPLE II.

161. *Let the Solid, whoſe Surface you would find, be a Sphere* AEBH.

In which Caſe, putting the Radius OH $=a$, AF $=x$, H*m* $=\dot{x}$, &c. we ſhall (by reaſon of the ſimilar Triangles OHF and H*mh* *) have y (FH) : a (OH) :: $\dot{x}$ (H*m*) : $\dot{z}$ (H*h*) $=\frac{a\dot{x}}{y}$: Therefore $\dot{w}$ $(2py\dot{z})=2pa\dot{x}$; and conſequently the Superficies (w) itſelf $=2pax=$ AF × *Periph.* AEBH. Which, if the whole Sphere be taken, will become AB × *Periph.* AEBH $=$ four times the Area BEAHO. * Art. 68.



Hence the Superficies of a Sphere is equal to four times the Area of its greateſt Circle: And the convex Superficies of any Segment thereof, is to that of the *Whole*, as the Axis (or Thickneſs) of the Segment to the Diameter of the Sphere.

EXAMPLE III.

162. *Wherein let the parabolic Conoid be propoſed.*

The Equation of the generating Parabola being $ax=y^2$, or $x=\frac{y^2}{a}$, we have $\dot{x}=\frac{2y\dot{y}}{a}$, and therefore $\dot{z}\left(=\sqrt{\dot{y}^2+\dot{x}^2}\,\dagger\right)=\sqrt{\dot{y}^2+\frac{4y^2\dot{y}^2}{a}}=\frac{\dot{y}\sqrt{a^2+4y^2}}{a}$: † Art. 135.

Hence $\dot{w}$ $(2py\dot{z})=\frac{2py\dot{y}}{a}\times\overline{a^2+4y^2}\big|^{\frac{1}{2}}$; whereof the

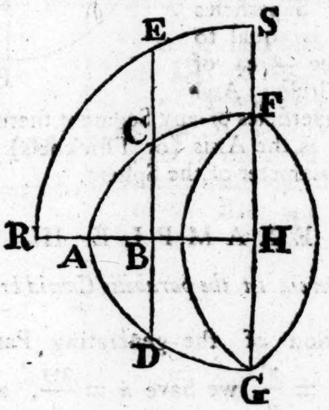
Fluent is $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a}$; which corrected (by ſup-

* Art. 79. poſing $y = 0$ *) gives $\frac{p\times\overline{a^2+4yy}|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$, for the Superficies ſought.

EXAMPLE IV.

163. *Let it be required to determine the Superficies of a Spheroid.*

Let ACFHG repreſent one half of the propoſed Spheroid, generated by the Rotation of the Semi-ellipſis FAG, about its Axis AH; put $AH = a$, FH (or HG) $= c$, $BH = x$, $BC = y$, $FC = z$, and the Superficies generated by FC (or GD) $= w$: Then, from the Na-



ture of the Ellipſis, we have $y = \frac{c}{a}\sqrt{a^2 - x^2}$; whence

† Art. 135. $\dot{y} = -\frac{cx\dot{x}}{a\sqrt{a^2 - x^2}}$, and conſequently $\dot{z}$ $(= \sqrt{\dot{x}^2 + \dot{y}^2}$ †)

=

$= \sqrt{\dot{x}^2 + \frac{c^2x^2\dot{x}^2}{a^2 \times \overline{a^2 - x^2}}} = \frac{\dot{x}\sqrt{a^4 - \overline{aa - cc} \times xx}}{a\sqrt{aa - xx}} =$ $\frac{\dot{x}\sqrt{a^4 - b^2x^2}}{a\sqrt{a^2 - x^2}} =$ (by putting (the Excentricity) $\sqrt{a^2 - c^2} = b) = \frac{b\dot{x}\sqrt{\frac{a^4}{bb} - x^2}}{a\sqrt{a^2 - x^2}}$: Therefore, in this Caſe, $\dot{w}$ $(2py\dot{x}) = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{bb} - x^2}$; whoſe Fluent, in an Infinite Series, is $2pcx \times \overline{1 - \frac{b^2x^2}{2.3a^4} - \frac{b^4x^4}{2.4.5a^8} - \frac{3b^6x^6}{2.4.6.7a^{12}}}$. But the ſame Fluent may be, *otherwiſe*, very eaſily exhibited by means of the Area of a Circle: For, if from the Center H, with a Radius equal to $\frac{aa}{b}$, a Circle SER be deſcribed, and the Ordinate BC be produced to interſect it in E, it is evident that BE $= \sqrt{\frac{a^4}{bb} - xx}$, and that the Fluxion of the Area ESHB will be expreſſed by $\dot{x}$ $\sqrt{\frac{a^4}{bb} - x^2}$; which being to $\frac{2pbc\dot{x}}{aa} \times \sqrt{\frac{a^4}{bb} - x^2}$, the Fluxion before found, in the conſtant Ratio of 1 to $\frac{2pbc}{a^2}$, their Fluents muſt therefore be in the ſame Ratio; and ſo the latter, expreſſing the Superficies CFGD, will conſequently be $= \frac{2pbc}{aa} \times \text{BESFH} = 2p \times \frac{\text{FH}}{\text{HS}}$ $\times$ BESFH.

This Solution, it may be obſerved, obtains only in Caſe of an *oblong* Spheroid, generated by the Rotation of the Ellipſis about its greater Axis; for, in an *oblate*

Spheroid, generated about the leſſer Axis, the Value of b $(\sqrt{a^2 - c^2})$ will be impoſſible; ſince, in this Caſe HF is greater than HA. But, if we, *here*, put $b = \sqrt{c^2 - a^2}$, and $d = \frac{a^2}{b}$, the Value of $\dot{w}$ (found above) will become $= \frac{2pbc\dot{x}}{a^2} \sqrt{\frac{a^4}{bb} + x^2} = \frac{2pc\dot{x}}{d} \sqrt{d^2 + x^2} = \frac{2pc}{d} \times \dot{x} \sqrt{a^2 + x^2}$: Whoſe Fluent may be brought out by help of a Table of Logarithms: For, let the variable Part $\dot{x} \sqrt{d^2 + x^2}$ be transformed to $\left(\frac{\dot{x} \times \overline{d^2 + x^2}}{\sqrt{d^2 + x^2}} = \frac{d^2\dot{x} + x^2\dot{x}}{\sqrt{d^2 + x^2}} = \frac{d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} = \right) \frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} + \frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, ſo that the Numerator of the firſt Term $\frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}}$ (now in a given Ratio to the Fluxion of the Quantity under the radical Sign) may be had by the common Rule *; by which means we get $\frac{1}{2}\sqrt{d^2x^2 + x^4}$, for the true Fluent of the ſaid Term; to which adding the Fluent of the other Term $\frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, or $\frac{\frac{1}{2}d^2\dot{x}}{\sqrt{d^2 + x^2}}$ (given by *Art.* 126.) there ariſes $\frac{1}{2}x\sqrt{d^2 + x^2} + \frac{1}{2}d^2 \times$ hyp. Log. $x + \sqrt{d^2 + x^2}$, for the Fluent of $\dot{x}\sqrt{d^2 + x^2}$: And this, corrected † and multiplied by $\frac{2pc}{d}$, gives $\frac{pcx}{d} \sqrt{d^2 + x^2} + pcd \times$ hyp. Log. $\frac{x + \sqrt{dd + xx}}{d}$, for the Superficies in this Caſe, where the propoſed Spheroid is an oblate One.

* Art. 77.

† Art. 78.

E X-

EXAMPLE V.

164. *Let the Solid, whoſe Superficies is ſought, be the hyperbolical Conoid.*

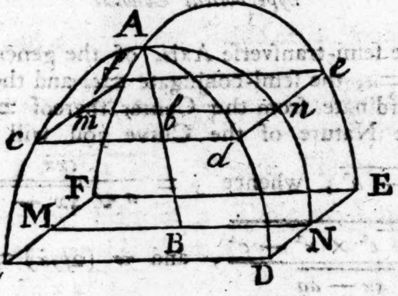
Let the ſemi-tranſverſe Axis, of the generating Hyperbola, $=a$, the ſemi-conjugate $=c$, and the Diſtance of any Ordinate from the Center thereof $=x$; then from the Nature of the Curve you will have $y = \frac{c}{a}\sqrt{x^2 - a^2}$; whence $\dot{y} = \frac{c x \dot{x}}{a\sqrt{xx - aa}}$, $\dot{z} = \frac{\dot{x}\sqrt{\overline{a^2+c^2} \times x^2 - a^4}}{a\sqrt{xx - aa}}$, and $\dot{w}$ $(2py\dot{z}) = \frac{2pc\dot{x}}{aa} \times \sqrt{\overline{aa+cc} \times xx - a^4}$; which laſt Value, if d^2 be put $= \frac{a^4}{a^2+c^2}$, will be more commodiouſly expreſſed by $\frac{2pc\dot{x}}{d}\sqrt{x^2 - d^2}$: whereof the Fluent, by proceeding as in the latter Part of the foregoing Example, will come out $= \frac{pcx\sqrt{xx - dd}}{d} - pcd \times$ *hyp. Log.* $x + \sqrt{x^2 - d^2}$: Which corrected (by taking $x=a$) becomes $\frac{pcx}{d}\sqrt{xx - dd} - pc^2, - pcd \times$ *hyp. Log.* $\frac{x + \sqrt{x^2 - d^2}}{a + \frac{cd}{a}}$, the true Meaſure of the required Superficies.

EXAMPLE VI.

165. *Let it be propoſed to find the Superficies of the Solid called a Groin.* (Vid. Art. 155.)

Let $bcdef$ be any Section of the Solid parallel to the Baſe thereof, and let x denote its Diſtance from the

Vertex A, alſo put z equal to the correſponding Arch A*n* of the ſemi-circular Section N*n*A *&c.* whoſe Radius AB or BN let be denoted by *a*.



It appears from *Art.* 161. that $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$:

* Art. 159. Which Value, multiplied by $(2\sqrt{2ax - xx})$ that of *de* $(= 2bn)$ gives $2a\dot{x}$ * for the Fluxion of one of the four equal convex Superficies by which the Solid is bounded. Hence the whole Superficies (excluding the Baſe) comes out $= 8a^2$: Which therefore is exactly equal to twice the Baſe.

If the Solid be ſuppoſed a Groin of any other Kind, ſuch that its two equal Sections, through the Middle of the oppoſite Sides, are other Curves than Circles, the Superficies may *ſtill* be had in the ſame manner; and will be always in proportion to the Superficies ariſing from the Revolution of either of the ſaid equal Curves about its Axis, as a Square is to its inſcribed Circle. Thus, the Superficies of a parabolic Conoid being $=$

$\frac{p \times \overline{aa + 4yy}|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$ (*by Art.* 162.) the convex Superficies of the Groin, ſuppoſing the generating Curve A*n*N to be a Parabola, will therefore be $=$

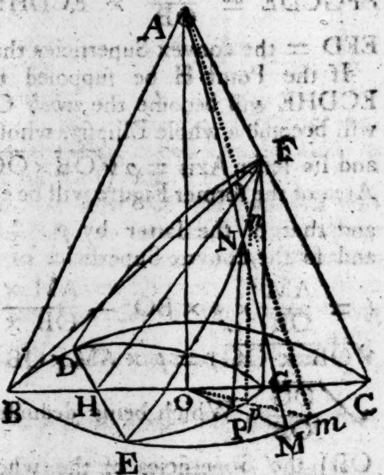
$\frac{4 \times \overline{a^2 + 4yy}|^{\frac{3}{2}}}{6a} - \frac{4a^2}{6}$.

EX-

EXAMPLE VII.

166. *Wherein let it be required to find the convex Superficies of a conical* Ungula ECFD; formed by a Plane DFE paſſing through the Baſe of the Cone.

Let a right-angled Triangle AOM (whoſe Baſe OM is the Radius of the Circle BDCE) be ſuppoſed to revolve about the Axis AO; whilſt a Right-line NP, drawn perpendicular to OM from the interſection of AM and the Arch EFD, traces out, upon the Baſe of the Cone, the Curve-line EPGD.



If MPOAN and *mp*OA*n* be conſidered as two Poſitions of the generating Triangle indefinitely near to each other, it is evident that the Space MA*m*, generated by AM, will be to the Space MO*m*, generated by OM, as AM to OM, or OB. Whence, MN and MP being proportional Parts of AM and OM (becauſe NP is parallel to AO) it is likewiſe plain that the Spaces MN*nm* and MP*pm*, generated by thoſe Parts, will be to each other in the ſame Ratio of AM to OB. And ſince this every where holds, it follows that the whole Space (ENM) *&c.* generated by MN, will be to that (EPM) generated by PM, as AM to OB: And ſo the whole required Superficies (generated by AM) is truly repreſented by $\frac{AM}{OB} \times$ Area EPGDCE.

But now, to find this Area, EPGDCE, it is obſervable that the Area of the Plane DFE (being the Segment of a Conic-ſection) is given, *by Art.* 115. 129 or 130. And it is very eaſy to apprehend and demonſtrate that the Area ſo given will be to that of EGDH, as the Radius to the Co-ſine of the Angle of the Inclination of the ſaid Plane to the Baſe, or as HF to HG. Therefore, ſeeing EGDH is $= \frac{HG}{HF} \times EFD$, we have EPGDCE $(= ECDHE - EGDH) = ECDHE - \frac{HG}{HF} \times EFD$; and conſequently $\frac{AM}{OB} \times EPGCDE = \frac{AM}{OB} \times ECDHE - \frac{AM \times HG}{OB \times HF} \times EFD =$ the convex Superficies that was to be found.

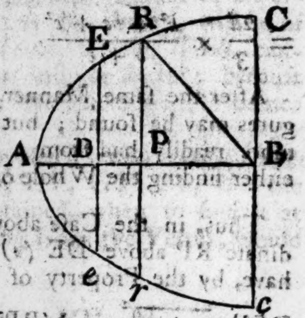
If the Point H be ſuppoſed to coincide with B, ECDHE will become the *whole* Circle CB; and EDF will become a whole Ellipſis, whoſe greater Axis is BF, and its leſſer Axis $= 2\sqrt{OB \times OG}$.* Therefore, the Area of the former Figure will be expreſſed by $p \times BO^2$†, and that of the latter by $p \times \frac{1}{2} BF \times \sqrt{OB \times OG}$; and ſo the convex Superficies of the Part BFC will be $(= \frac{AM}{OB} \times p \times BO^2 - \frac{AM \times BG}{OB \times BF} \times p \times \frac{1}{2} BF \times \sqrt{OB \times OG}) = p \times AM \times OB - p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$: Which being deducted from $(p \times AM \times OB)$ the Superficies of the whole Cone BAC, there reſts $p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$, for the Superficies of the oblique Cone BAF; which from hence is alſo given.

* Art. 41.

† Art. 124.

SCHOLIUM.

167. In moſt of the Examples, delivered in the four laſt Sections, the Part of the propoſed *Figure* next the Vertex, whether, a Curve, Solid, or Superficies, is firſt found; from whence, by taking the Altitude (x) of that Part equal to (a) the Altitude given, the Content of the *Whole* is deduced: But, if the Content of the lower Segment (BCED) of any Figure (ABC) ariſing by taking away a Part (ADE) next the Vertex, be required; then the Difference between the *Whole* and the Part taken away (found as before explained) will be the Quantity ſought.



Thus, for Example, let ABC be the common *Parabola*, and let it be propoſed to find the Content of the Part, BCED, included between any two Ordinates BC (b) and DE (c) at a given Diſtance BD (d) from each other: Then, the Equation of the Curve being $ax = y^2$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $y\dot{x}$ * $= \frac{2y^2\dot{y}}{a}$, whoſe Fluent $\frac{2y^3}{3a}$ is a general Expreſſion for the Area comprehended between the Vertex and the Ordinate y: Whence, expounding y, by b and c ſucceſſively, we get $\frac{2b^3}{3a}$ and $\frac{2c^3}{3a}$ for the correſponding Values of ABC and ADE; whoſe Difference $\frac{2b^3 - 2c^3}{3a}$ is the required Area BCED: But, to expreſs the ſame independent of a, it will be, by the Property of the Curve, $b^2 : c^2 :: \text{AB} : \text{AD}$; whence,

* Art. 112.

whence, by Diviſion, $b^2 : b^2 - c^2 :: AB : BD$ *(d)* and conſequently $\frac{b^2-c^2}{d} = \frac{b^2}{AB} = a$; which firſt Value being wrote inſtead of *a*, there reſults $BCED = \frac{\overline{2b^3 - 2c^3} \times d}{3b^2 - 3c^2}$ $= \frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$.

After the ſame Manner, the Segments of other Figures may be found; but in many Caſes they will be more readily had from a direct Inveſtigation, without either finding the Whole or the Part taken away.

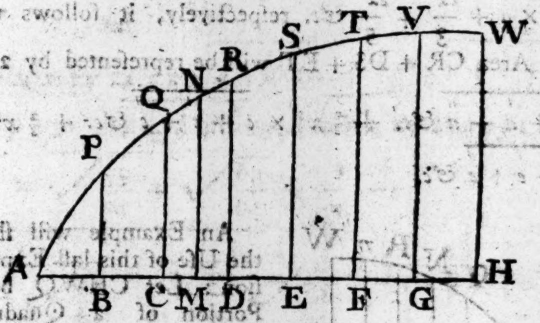
Thus, in the Caſe above, if the Exceſs of any Ordinate RP above DE *(c)* be denoted by *w*, we ſhall have, by the Property of the Curve, $b^2 - c^2$ $(BC^2 - DE^2) : \overline{c + w}|^2 - c^2$ $(RP^2 - DE^2) :: DB$ *(d)* $: DP = \frac{d \times \overline{2cw + w^2}}{b^2 - c^2}$, whoſe Fluxion $\left(d \times \frac{2c\dot{w} + 2w\dot{w}}{b^2 - c^2}\right)$ multiplied by $c + w$ $(= PR)$ gives $d \times \frac{2c^2\dot{w} + 4cw\dot{w} + 2w^2\dot{w}}{b^2 - c^2}$, for the Fluxion of the Area DPRE: Whereof the Fluent $\left(\text{which is } 2dw \times \frac{c^2 + cw + \frac{1}{3}w^2}{b^2 - c^2}\right)$ will, when $w = b - c$ (or $RP = BC$) be truly expounded by $\frac{2d \times \overline{b - c} \times \overline{\frac{1}{3}b^2 + \frac{1}{3}bc + \frac{1}{5}c^2}}{b^2 - c^2}$ or its Equal, $\frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$; *the ſame as before.*

Again, for another Example, let CED*ec* be conſidered as the lower Fruſtrum of an Hemiſphere, whoſe Center is the Point B: Then, BP being *here*, denoted by *w*, we ſhall have y^2 $(= BR^2 - BP^2) = b^2 - w^2$,
* Art. 145. and conſequently $py^2\dot{w}$ * $= p \times \overline{b^2\dot{w} - w^2\dot{w}}$; whoſe Fluent

Fluent $(p \times \overline{b^2w - \frac{1}{3}w^3} = \frac{1}{3}pw \times \overline{3b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + y^2} = \frac{1}{3}p \times BP \times \overline{2BC^2 + PR^2})$ is the true Content of the Part CED*ec*; which will alſo hold when the Figure is a Spheroid.

This laſt Method, of finding the Content of a Portion of a Figure, remote from the Vertex, will be of Service, when the general Value, for the *Whole*, cannot be expreſſed without an infinite Series; becauſe ſuch a Series, in that Caſe, not converging, becomes uſeleſs *.

By dividing the whole propoſed Figure, AHW, into a Number of ſuch Portions, HV, GT, FS, *&c.* the Content thereof may be obtained, when to find it at once, by a Series, commencing from the Vertex, would be altogether impracticable.

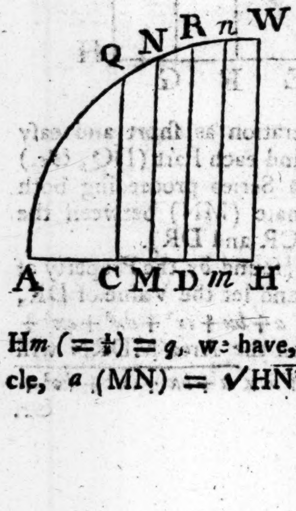


But, to render ſuch an Operation as ſhort and eaſy as may be, it will be proper to find each Part (DQ, *&c.*) of the Figure, by means of a Series proceeding both Ways, from the middle Ordinate (MN) between the two correſponding Extremes (CR and DR).

Thus, let the Value of MN (found by the Property of the Curve) be denoted by a; and let the Value of DR, in a Series, be repreſented by $a + bx + cx^2 + dx^3 + ex^4 + fx^5 +$ *&c.* where $x = MD$; then the Area MDRN will be repreſented by the Fluent of $\dot{x} \times \overline{a + bx + cx^2 + dx^3 +}$

* Art. 93.

&c. or by $x \times \overline{a + \frac{bx}{2} + \frac{cx^2}{3} + \frac{dx^3}{4} + \text{\&c.}}$ And by writing $-x$ inſtead of x, the Ordinate CQ will be expreſſed by $a - bx + cx^2 - dx^3$ *&c.* and the Area MCQN, by $x \times \overline{a - \frac{bx}{2} + \frac{cx^2}{3} - \frac{dx^3}{4} + \frac{ex^4}{5}}$ *&c.* whence the Area CDRQ is $= 2x \times \overline{a + \frac{cx^2}{3} + \frac{ex^4}{5} + \frac{gx^6}{7} + \text{\&c.}}$ Therefore, if DE, EF, FG, and GH be ſuppoſed, each, $=$ BC $(2x)$ and the Areas DS, ET, *&c.* (found as above) be denoted by $2x \times \overline{\overset{\prime}{a} + \frac{\overset{\prime}{c}x^2}{3} + \frac{\overset{\prime}{e}x^4}{5}}$ *&c.* and $2x \times \overline{\overset{\prime\prime}{a} + \frac{\overset{\prime\prime}{c}x^2}{3} + \frac{\overset{\prime}{e}x^4}{5}}$ *&c.* reſpectively, it follows that the Area CR + DS + ET will be repreſented by $2x \times \overline{a + \overset{\prime}{a} + \overset{\prime\prime}{a}}$ *&c.* $+ \frac{2}{3}x^3 \times \overline{c + \overset{\prime}{c} + \overset{\prime\prime}{c}}$ *&c.* $+ \frac{2}{3}x^5 \times \overline{e + \overset{\prime}{e} + \overset{\prime\prime}{e}}$ *&c.*



An Example will ſhew the Uſe of this laſt Expreſſion: Let CHWQ be a Portion of a Quadrant HAW of a Circle, whoſe Baſe HC (conceived to be divided into four equal Parts) is equal half the Radius AH, repreſented by *Unity*. Then, putting CM $(= \text{DM} = \text{D}m = m\text{H} = \frac{1}{8}) = x$, HM $(= \frac{3}{8}) = p$, and Hm $(= \frac{1}{8}) = q$, we have, by the Property of the Circle, a (MN) $= \sqrt{\text{HN}^2 - \text{HM}^2} = \sqrt{1 - pp}$, and

DR $(=\sqrt{HR^2-HD^2}) = \sqrt{1-\overline{p-x}^2} = \sqrt{1-p^2+2px-x^2} = \sqrt{a^2+2px-x^2}$; which, in a Series, is $\left(=a+\frac{2px-x^2}{2a}-\frac{\overline{2px-x^2}|^2}{8a^3}+\right.$ &c.$)$ $=a+\frac{px}{a}-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}\times x^2$ &c. Therefore, in this Caſe, $b=\frac{p}{a}$, $c=-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}$, &c. Which Value of c, by writing $1-a^2$ for its Equal p^2, will be reduced to $-\frac{1}{2a^3}$. From whence it is alſo evident that $\overset{\prime}{c}=-\frac{1}{2\overset{\prime}{a}{}^3}$ (ſuppoſing $\overset{\prime}{a}$ (mn) $=\sqrt{1-q^2}$)

Conſequently $2x\times\overline{a+\overset{\prime}{a}+\overset{\prime\prime}{a}\text{ \&c.}}+\frac{2}{3}x^3\times\overline{c+\overset{\prime}{c}+\overset{\prime\prime}{c}\text{ \&c.}}+\frac{2}{5}x^5\times\overline{e+\overset{\prime}{e}+\overset{\prime\prime}{e}\text{ \&c.}}$ $\left(=\overline{a+\overset{\prime}{a}}\times 2x+\overline{c+\overset{\prime}{c}}\times\frac{2x^3}{3}\right)=\overline{\sqrt{\frac{55}{64}}+\sqrt{\frac{63}{64}}}\times\frac{1}{4}-\overline{\frac{1}{\frac{2\times55}{64}\sqrt{\frac{55}{64}}}+\frac{1}{\frac{2\times63}{64}\sqrt{\frac{63}{64}}}}\times\frac{2}{64\times8\times3}=$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{1}{3\times55\sqrt{55}}-\frac{1}{3\times63\sqrt{63}}=$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{\sqrt{55}}{3\times55\times55}-\frac{\sqrt{63}}{3\times63\times63}=$$

0,48730 = the Area, CHWQ, that was to be found.

This Example, choſen as an Illuſtration of the foregoing Method, may indeed be wrought the common Way; whence the very ſame Concluſion is brought out

(*Vide Art.* 124.) But that Method is alſo applicable to any other Caſe, whether the Part propoſed be near to the Vertex, or remote from it; and whether the Figure itſelf be a Curve, Solid or Superficies; ſince the Meaſure thereof may, always, be expreſſed by the Area of a Curve.

There is another Way, well known to Mathematicians, whereby the Area of a Curve may be determined, by means of a Number of equidiſtant Ordinates; which Method, derived from *that of Differences*, may, alſo, be uſed to good Purpoſe, in Caſes like thoſe above ſpecified: But, it having been treated of by ſeveral *others*, and alſo in my *Diſſertations*, the Reader will excuſe me, if no further Notice is taken of it here.

SECTION XI.

Of the Uſe of FLUXIONS *in finding the Centers of Gravity, Percuſſion, and Oſcillation of Bodies.*

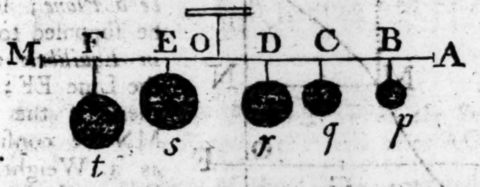
168. THE Center of Gravity is that Point of a Body, by which, if it were ſuſpended, it would reſt *in Equilibrio*, in any Poſition.

LEMMA.

169. *Let p, q, r, s, &c. be any Number of given Weights, hanging at an inflexible Line (or Rod)* AM *ſuſpended* in Equilibrio, *in an horizontal Poſition, at the Point* O; *to determine the Poſition of that Point.*

Since *(by Mechanics)* the Force of any Weight *(p)* to raiſe the oppoſite End (M) of the Balance, is as that Weight drawn into its Diſtance (BO) from the Fulcrum,

crum, we ſhall, from the Equality of theſe Forces, have $p \times OB + q \times OC + r \times OD = s \times OE + t \times OF$,



that is $p \times \overline{AO - AB} + q \times \overline{AO - AC} + r \times \overline{AO - AD} = s \times \overline{AE - AO} + t \times \overline{AF - AO}$, and conſequently $AO = \dfrac{p \times AB + q \times AC + r \times AD + s \times AE + t \times AF}{p + q + r + s + t}$.

From which it appears, that, *if each Weight be multiply'd by its Diſtance from the End (or any given Point) of the Axis, the Sum of all the Products divided by the Sum of all the Weights, will give the Diſtance of the Center of Gravity from that End (or Point.)*

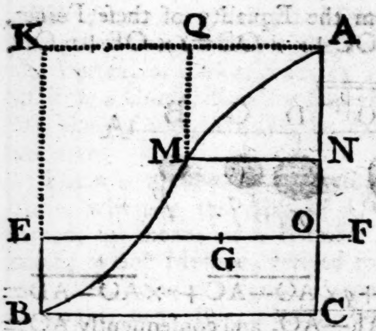
Note. The Products here mentioned are, uſually, call'd *the Forces*, of their reſpective Weights; not in reſpect to their Action at the Center O (which is expreſſed by a different Quantity) but with regard to the Effects they have in the Concluſion, or the Value of AO; which appear to be in that Ratio.

PROPOSITION I.

170. *To determine the Center of Gravity of a Line, Plane, Superficies, or Solid* (admitting the three former capable of being affected by Gravity.)

Let AMBC be the propoſed Figure, and G the Center of Gravity thereof; thro' which, parallel to the Horizon, let the Line EF be drawn, interſecting AC, at Right-angles, in O; alſo let AK and NM be perpendicular to AC, and parallel to EF.

171. Caſe



171. Caſe 1. *If the Figure* AMBC *be a Plane*; let it be ſuppoſed to reſt *in Equilibrio* upon the Line EF; and then, if the Line MN be conſider'd as a Weight, its Force (*defined above*) will be expreſſed by MN drawn into its Diſtance (AN) from the End of the Axis AC; that is by yx (ſuppoſing, as uſual, AN$=x$ and MN$=y$.) This, therefore, multiply'd by the Fluxion of AN, gives $yx\dot{x}$ for the Fluxion of the Force of the Plane AMN; whoſe Fluent, when $x=$AC, expreſſes the Force of the whole Plane, or the Sum of all the Products of the Ordinates (or Weights) by their reſpective Diſtances from AK: Which Fluent being, therefore, divided by the Area ABC, or the Fluent of $y\dot{x}$ (according to the foregoing Lemma) the Quotient $\left(\frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}}\right)$ will give (AO) the Diſtance of the Center of Gravity from the Line AK.

172. Caſe 2. *If the Figure be a Solid*; let MN be a Section thereof by a Plane perpendicular to the Horizon; then, the Area of that Section being denoted by *A*, the Force thereof (conſidered as above) will be expreſſed by Ax, and the Fluxion of the Force of the Solid AMN by $Ax\dot{x}$; whoſe Fluent, divided by the Content of the Body, or the Fluent of $A\dot{x}$, gives AO, in this Caſe. But, if the Solid be the half (or the whole) of that ariſing from the Rotation of a Curve AMB about its Axis AC; then (putting p for the Area of the Circle whoſe Radius is Unity) A will become $= \frac{1}{2}py^2$ *; and conſequently AO $= \frac{Flu.\ \frac{1}{2}py^2x\dot{x}}{Flu.\ \frac{1}{2}py^2\dot{x}} = \frac{Flu.\ y^2x\dot{x}}{Flu.\ y^2\dot{x}}$.

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* Art. 145.

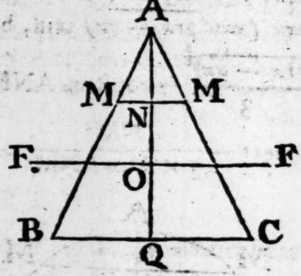
173. Caſe 3. *If the Figure propoſed be the Curve-line* AMB; then, the Force of a Particle at M being expreſſed by AN or MQ (x) we ſhall (putting AM $= z$) have $\frac{\text{Flu. } x\dot{z}}{z} =$ AO.

174. Caſe 4. *But if the Figure given be the Superficies generated by the Rotation of* AMB *about* AC.

Then, the Periphery of the Circle generated by the Point M being $= 2py$, it follows that $\frac{\text{Flu. } 2pyx\dot{z}}{\text{Flu. } 2py\dot{z}} = \frac{\text{Flu. } yx\dot{z}}{\text{Flu. } y\dot{z}} =$ AO.

EXAMPLE I.

175. *Let the Figure propoſed be the iſoſceles Triangle* ABC.



It is evident the Center of Gravity (O) will be ſomewhere in the Perpendicular AQ: And, if AQ$=a$, BC$=b$, AN $=x$, and MM$=y$; then y being $= \frac{bx}{a}$, we ſhall have, *by Caſe* 1, AO $\left(= \frac{\text{Flu. } yx\dot{x}}{\text{Flu. } y\dot{x}}\right) = \frac{\text{Flu. } x^2\dot{x}}{\text{Flu. } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2x}{3} = \frac{2}{3}$ AQ, when $x =$ AQ; and conſequently OQ $= \frac{\text{AQ}}{3}$.

In the very ſame manner, the Center of Gravity of any other (plane) Triangle will appear to be at $\frac{1}{3}$ of the Altitude of the Triangle.

EX-

EXAMPLE II.

176. *Let the Figure propoſed be a Parabola of any Kind;*

whereof the Equation is $y = \frac{x^n}{a^{n-1}}$.

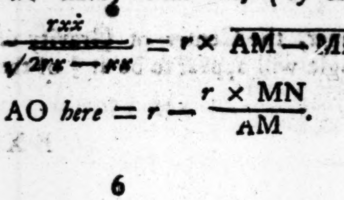
Here, $\frac{\text{Flu. } yx\dot{x}}{\text{Flu. } y\dot{x}} = \frac{\text{Flu. } x^{n+1}\dot{x}}{\text{Flu. } x^n\dot{x}} = \frac{n+1}{n+2} \times x =$ the Diſtance of the Center of Gravity from the Vertex of the Curve.

EXAMPLE III.

177. *Let* BAC *be a Segment of a Circle.*

Then, if the Radius thereof be put $= r$, we ſhall have y (NM) $= \sqrt{2rx - xx}$: Whence the Fluent of $yx\dot{x}$ $(x\dot{x}\sqrt{2rx - xx})$ will, by *Art.* 163. be found $= -\frac{\overline{2rx - xx}|^{\frac{3}{2}}}{3} + r \times$ Area ANM; which divided by ANM, gives $r - \frac{\text{NM}^3}{3 \times \text{Area ANM}} =$ AO*. This, therefore, when

* Art. 171.



BAC is a Semi-circle, becomes $= \frac{576}{1000} \times r$, nearly.

But, with reſpect to the Center of Gravity of the Arch BAC; we have, *Flu.* $x\dot{z}$, *(by Caſe* 3.*)* $=$ Fluent of $\frac{rx\dot{x}}{\sqrt{2rx - xx}} = r \times \overline{\text{AM} - \text{MN}}$; and conſequently AO *here* $= r - \frac{r \times \text{MN}}{\text{AM}}$.

EXAMPLE IV.

178. *Let* ABC (ſee the preceding Figure) *repreſent a Segment of a Sphere, or Spheroid.*

In which Caſe, denoting the Axis of the Sphere, or Spheroid, by a, and the other Axis of the generating Curve, when an Ellipſis, by b, we have $y^2 = \frac{bb}{aa} \times \overline{ax - xx}$;

and therefore $\frac{Flu.\ y^2 x\dot{x}}{Flu.\ y^2 \dot{x}}{}^* = \frac{Flu.\ \overline{ax-xx} \times x\dot{x}}{Flu.\ \overline{ax-xx} \times \dot{x}} =$ * Art. 172.

$$= \frac{\frac{1}{3}ax^3 - \frac{1}{4}x^4}{\frac{1}{2}ax^2 - \frac{1}{3}x^3} = \frac{\frac{1}{3}ax - \frac{1}{4}x^2}{\frac{1}{2}a - \frac{1}{3}x} = \frac{x \times \overline{4a - 3x}}{6a - 4x} = AO.$$

If the Solid be an hyperbolical Conoid, the Diſtance (AO) of its Center of Gravity from the Vertex, will alſo be exhibited by the Expreſſion here brought out, when the negative Signs are changed to poſitive ones.

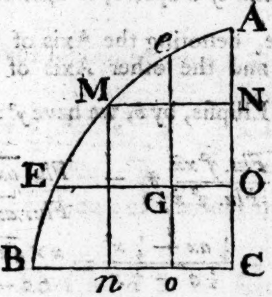
179. In thoſe Caſes where the Figure cannot be divided into two Parts, equal and like to each other (as a Curve is by its Axis, *&c.*) the Poſition of two Lines EO, *eo* (*ſee the enſuing Figure*) muſt be determined, *as above*; in whoſe Interſection (G) the Center of Gravity will be found.

EXAMPLE V.

Let ABC *be a Semi-parabola of any Kind*; whereof the

Equation is $y = \frac{x^n}{a^{n-1}}$.

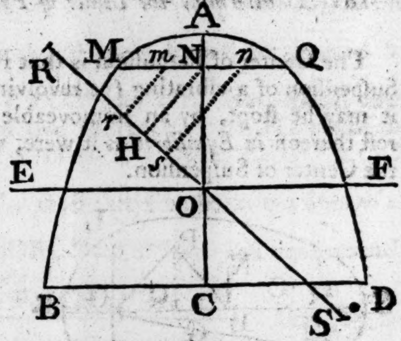
It appears, *from Ex.* 2. that (AO) the Diſtance of EGO from the Vertex, is expreſſed by $\frac{n+1}{n+2} \times AC$: But to find the Poſition of *oGe* (perpendicular to EO) let M*n* be parallel to *eo*, or AC; then, AN being $= x$, and NM $(y) = \frac{x^n}{a^{n-1}}$; if AC be denoted by b, we



ſhall have M$n = b - x$, and M$n \times$ NM $\times \dot{y} = \overline{b - x} \times \frac{x^n}{a^{n-1}} \times \frac{nx^{n-1}\dot{x}}{a^{n-1}} = \frac{nbx^{2n-1}\dot{x} - nx^{2n}\dot{x}}{a^{2n-2}}$, for the Fluxion of the Sum of the Forces in this Caſe (*Vid. Art.* 171.) whoſe Fluent $\left(\frac{nbx^{2n}}{2na^{2n-2}} - \frac{nx^{2n+1}}{\overline{2n+1} \times a^{2n-2}}\right.$ $= \frac{x^{2n}}{a^{2n-2}} \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} = y^2 \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} =$ $y^2 \times \frac{b}{4n+2}$, or $\frac{\text{BC}^2 \times \text{AC}}{4n+2}$, when $x = b$) divided by the Area ABC $\left(= \frac{\text{BC} \times \text{AC}}{n+1}\right)$ gives $\frac{n+1}{4n+2} \times$ BC for the true Value of Co, or OG. Which, in caſe of the common Parabola, where $n = \frac{1}{2}$, and where AO $\left(\frac{n+1}{n+2} \times \text{AC}\right) = \frac{3}{5}$AC, will become $= \frac{3}{8}$CB.

Before I leave this Subject it may not be improper to take notice, *that*, whatever Line you found your Calculations upon, by ſuppoſing the Figure to reſt, *in Equilibrio*,

Equilibrio, upon that Line, the very ſame Point, for the Place of the Center of Gravity, will be determined.



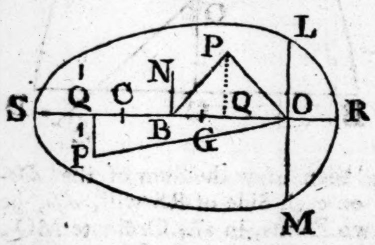
180. Thus, let O be the Point in the Axis AC, of a given Curve BAD, determined, as above, by ſuppoſing the Figure to reſt upon EF perpendicular to AC; and let RS be any other Line paſſing through the Point O; then I ſay the Sum of the *Momenta* of the Particles on each Side of RS will, *alſo*, be equal. For, if from two Points, in any Ordinate MQ, equally diſtant from the middle Point N, two Perpendiculars *mr* and *ns* be let fall upon RS, the Efficacy of thoſe two Points, in reſpect to RS, will be repreſented by $mr + ns$, or its Equal 2NH (ſuppoſing NH alſo perpendicular to RS.) Whence the Efficacy of all the Particles in MQ, will be expreſſed by their Number multiplied by NH, or by MQ × NH: Which is to their Efficacy (MQ × ON) when referred to the Line EF, in the conſtant Ratio of NH to ON, or of the Sine of the Angle RON to Radius. Whence it is evident that the Force of all the Ordinates (or the whole Curve) in the former Caſe, muſt be to that in the latter, in the ſame Ratio: But the ſaid Force, in the one Caſe, is equal to nothing by Hypotheſis, therefore it muſt be likewiſe ſo in the other: And conſequently the Sum of the *Momenta* of the Particles, on each Side of RS, equal to each other.

Much after the ſame manner the thing may be proved, in a Solid: Whence it will appear that there is actually ſuch a (fixed) Point in a Body as the Center of Gravity is defined to be; Which, however evident from mechanical Conſiderations, is not ſo eaſy to demonſtrate, *geometrically*, from the Reſolution of Forces.

PROPOSITION II.

181. *To determine the Center of Percuſſion of a Body.*

The Center of Percuſſion is that Point, in the Axis of Suſpenſion of a vibrating (or revolving) Body, at which it may be ſtopt, by an immoveable Obſtacle, ſo as to reſt thereon *in Equilibrio* as it were, without acting upon the Center of Suſpenſion.



Let O be the Point of Suſpenſion, G the Center of Gravity, and SLM a Section of the Body, by the Plane wherein the Axis of Suſpenſion OGS performs its Motion; to which Section let all the Particles of the Body be conceived to be transferred in ſuch Parts thereof where they would be projected into *(orthographically)* by Lines parallel to the Axis of Motion; which Suppoſition will neither affect the Place of the Center of Gravity nor the angular Motion of the Body.

Since the angular Velocity of any Particle P is as the Diſtance, or Radius, OP, its Force in the Direction, PB, perpendicular to OP, will be expreſſed by $P \times OP$. Therefore the Efficacy of that Force upon the Axis, at B, in the perpendicular Direction BN (ſuppoſing the Axis ſtopt at C the Center of Percuſſion) will be $P \times OP \times \frac{OP}{OB}$, whoſe Power to turn the Body about the Point C is therefore as $P \times OP \times \frac{OP}{OB} \times BC = P \times \frac{OP^2 \times BC}{OB} = P \times \frac{OP^2 \times \overline{OC - OB}}{OB} = P \times \frac{OP^2 \times OC}{OB} - P \times OP^2$; which, if PQ be made Perpendicular to

OS, will at laſt (becauſe $\frac{OP^2}{OB} = OQ$) be reduced to $P \times OQ \times OC - P \times OP^2$. By the very ſame Argument, the Force of any other Particle $\acute{P}$ will be denoted by $\acute{P} \times O\acute{Q} \times OC - \acute{P} \times O\acute{P}^2$ *&c. &c.* But, as all theſe Forces muſt deſtroy one another (by the Nature of the Problem) the Sum of all the Quantities $P \times OQ \times OC$, $\acute{P} \times O\acute{Q} \times OC$, *&c.* muſt therefore be = the Sum of all the Quantities $P \times OP^2$, $\acute{P} \times O\acute{P}^2$ *&c.* and conſequently

$$OC = \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{P \times OQ + \acute{P} \times O\acute{Q} + \&c.\ \&c.}.$$

But (by the preceding Propoſition) the Sum of all the Quantities $P \times OQ + \acute{P} \times O\acute{Q} +$ *&c.* is equal to OG × by the Content of the Body. Therefore OC is likewiſe =

$$\frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{OG \times \textit{Body}}.$$

The ſame otherwiſe.

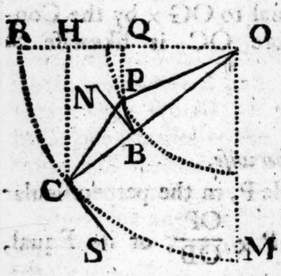
Since the Force of the Particle P, in the perpendicular Direction NB, is defined by $P \times \frac{OP^2}{OB}$, or its Equal, $P \times OQ$, the Sum of all the Quantities $P \times OQ$, $\acute{P} \times O\acute{Q}$, *&c. &c.* will expreſs the Force which, acting at C perpendicular to OS, is ſufficient to ſtop the Body, without the Center of Suſpenſion O being any way affected: This Sum, therefore, drawn into OC ($= OC \times P \times OQ + \acute{P} \times O\acute{Q} +$ *&c. &c.*) is as the Efficacy of the ſaid Force to turn the Body about the Point O. But the Force of the Particle P, in the Direction BN being $P \times \frac{OP^2}{OB}$, its Efficacy to turn the Body about the ſame

Point (the contrary way) is as $P \times OP^2$; and conſequently the Efficacy of all the Particles as the Sum of all the Quantities $P \times OP^2$, $\overset{\prime}{P} \times O\overset{\prime}{P}{}^2$ *&c. &c.* Therefore (Action and Re-action being equal) we have $OC \times \overline{P \times OQ + \overset{\prime}{P} \times O\overset{\prime}{Q}} + \text{\&c.} = P \times OP^2 + \overset{\prime}{P} \times O\overset{\prime}{P}{}^2 + \text{\&c.}$ *the ſame as before.*

For the Center of Oſcillation, it will be requiſite to premiſe the following

LEMMA.

182. *Suppoſe two exceeding ſmall Weights* C *and* P, *acting on each other by means of an inflexible Line (or Wire* PC) *to vibrate in a vertical Plane* ROPCM, *about the Center* O; *it is required to determine how much the Motion of the one is affected by the other.*



Let CH and PQ be perpendicular to the horizontal Line OR; alſo let PB and CS be perpendicular to OP and OC reſpectively.

If the Force of Gravity be denoted by Unity, the Forces acting in the Directions CS and PB, whereby the Weights, in their Deſcent, are accelerated, will, according to the Reſolution of Forces, be repreſented by $\frac{OH}{OC}$ and $\frac{OQ}{OP}$. Moreover, ſince the Velocities are always in the Ratio of the Radii OC and OP, if the foreſaid Forces were to be in that Ratio, or that of P was to become $\frac{OH}{OC} \times \frac{OP}{OC}$, inſtead of $\frac{OQ}{OP}$. I ſay, in that Caſe, it is plain the Weights would continue their Motion without

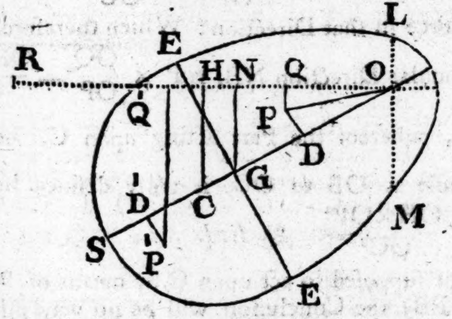
out affecting each other, or acting at all on the Line of Communication PC (or PB). Whence, the Exceſs of $\frac{OQ}{OP}$ above $\frac{OH}{OC} \times \frac{OP}{OC}$ muſt be the accelerative Force whereby the Weight P acts upon the Line (or Wire) OC, in the Direction PB; which multiply'd by the Weight P gives $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ for the abſolute Force in that Direction: Which therefore, in the perpendicular Direction NB, is $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}} \times \frac{OP}{OB}$; whereof the Part acting upon C, being to the Whole as OB to OC, is truly defined by $P \times \frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}$. *Q. E. I.*

If P be ſuppoſed to act upon C by means of PC (inſtead of PB) the Concluſion will be no way different: For, let F (to ſhorten the Operation) be put to denote the Force $(P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}})$ in the Direction PB, found above, then the Action thereof upon PC (according to the Principles of Mechanics) will be expreſſed by $F \times \frac{\textit{Radius}}{\textit{Co-ſ.}\ CPB}$; which therefore in the Direction SC, perpendicular to OC, is $F \times \frac{\textit{Radius}}{\textit{Co-ſ.}\ CPB} \times \frac{\textit{S.}\ PCO}{\textit{Radius}} = \frac{\textit{S.}\ PCO}{\textit{Co-ſ.}\ CPB} = \frac{\textit{S.}\ PCO}{\textit{S.}\ OPC} = F \times \frac{OP}{OC}$, *the very ſame as before.*

PROPOSITION III.

183. *To determine the Center of Oſcillation of a Body.*

The Center of Oſcillation is that Point, in the Axis (or Line) of Suſpenſion of a vibrating Body, into which if the whole Body was contracted, the angular Velocity and the Time of Vibration would remain unaltered.



Let LMS be a Section of the Body by a Plane, perpendicular to the Horizon and the Axis of Motion, paſſing thro' the Center of Gravity G and the Point of Suſpenſion O; and ſuppoſe all the Particles of the Body to be transferred to this Section, in ſuch Places of it, as they would be projected into (*orthographically*) by Perpendiculars falling thereon. (Which Suppoſition will no way affect the Concluſion, the angular Motion continuing the ſame.) Moreover let C be the Center of Oſcillation, or that Point in the Axis OS where a Particle of Matter (or a ſmall Weight) may be placed ſo as to be neither accelerated nor retarded by the Action of the other Particles of Matter ſituate in the Plane. Then, if, from C and any other Point P in the Plane LMS, two Perpendiculars CH and PQ be let fall upon the horizontal Line OR, the Force of a Particle (or Weight) at P to accelerate the Weight at C, will (*according to the foregoing Lemma*) be repreſented by P ×

$\overline{\frac{OQ}{OC} - \frac{OH\times OP^2}{OC^3}}$: Which, ſuppoſing GN perpendicular to OR, will alſo be expreſſed by $P\times\overline{\frac{OQ}{OC} - \frac{ON}{OG}\times\frac{OP^2}{OC^2}}$, or its Equal $P\times\frac{OQ\times OG\times OC - ON\times OP^2}{OC^2\times OG}$. In the very ſame manner the Force of any other Particle $\acute{P}$ will be repreſented by $\acute{P}\times\frac{O\acute{Q}\times OG\times OC - ON\times O\acute{P}^2}{OC^2\times OG}$ *&c. &c.*

Therefore the Forces of all the Particles deſtroying each other *(by Hypotheſis)* the Sum of all the Quantities $\overline{P\times OG\times OQ\times OC - ON\times OP^2}$ $+\overline{\acute{P}\times OG\times O\acute{Q}\times OC - ON\times O\acute{P}^2}$ *&c. &c.* muſt be equal to nothing: Whence $P\times OG\times OQ\times OC + \acute{P}\times OG\times O\acute{Q}\times OC$ *&c. &c.* $= P\times ON\times OP^2 + \acute{P}\times ON\times O\acute{P}^2$ *&c. &c.* and conſequently $OC = \frac{ON}{OG}\times\frac{P\times OP^2 + \acute{P}\times O\acute{P}^2 + \text{\&c.}}{P\times OQ + \acute{P}\times O\acute{Q} + \text{\&c.}}$. But *(by Art.* 171. and 172.) the Sum of all the Quantities $P\times OQ + \acute{P}\times O\acute{Q}$ *&c.* is equal to the Content of the Body multiplied by the Diſtance (ON) of the Center of Gravity G from the Line LM (perpendicular to OC); whence OC is alſo $= \frac{ON}{OG}\times\frac{P\times OP^2 + \acute{P}\times O\acute{P}^2\ \text{\&c.\&c.}}{ON\times Body} = \frac{P\times OP^2 + \acute{P}\times O\acute{P}^2\ \text{\&c.\&c.}}{OG\times Body}$ Which Expreſſion continuing the ſame in all Inclinations of the Axis OS, the Point C, thus determined is a fixed Point, agreeable to the Definition; and appears to be the ſame with the Center of Percuſſion; *ſee Art.* 181.

COROLLARY.

184. If PD, $\acute{P}\acute{D}$ *&c.* be perpendicular to OS, the Numerator of the Fraction found above, will become $P\times\overline{OG^2+GP^2-2OG\times GD}+\acute{P}\times\overline{OG^2+G\acute{P}^2+2OG\times G\acute{D}}$ + *&c. &c.* (ſince $OP^2=OG^2+GP^2-2OG\times GD$ *&c.*) Which, becauſe all the Quantities $P\times-2OG\times GD+\acute{P}\times 2OG\times G\acute{D}$ *&c.* or $P\times-GD+\acute{P}\times G\acute{D}$ *&c.* (by the Nature of the Center of Gravity) deſtroy one another, will be *barely* $=P\times\overline{OG^2+GP^2}+\acute{P}\times\overline{OG^2+G\acute{P}^2}$ + *&c. &c.* $=\overline{P+\acute{P}+}$ *&c.* $\times\, OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2$ + *&c.* = *Maſs* $\times\, OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2$ + *&c.* Whence it is evident that OC is, alſo,

$$\left(=\frac{\textit{Maſs}\times OG^2,+P\times GP^2+\acute{P}\times G\acute{P}^2+\textit{\&c. \&c.}}{\textit{Maſs}\times OG}\right)$$

$$=OG+\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\textit{\&c.}}{\textit{Maſs}\times OG};$$ and conſequently

$$CG=\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\textit{\&c. \&c.}}{\textit{Maſs}\times OG}.$$

Whence it appears that, *if a Body be turned about its Center of Gravity, in a Direction, perpendicular to the Axis of the Motion, the Place of the Center of Oſcillation will remain unaltered*; becauſe the Quantities $P\times GP^2$, $\acute{P}\times G\acute{P}^2$ are no way affected by ſuch a Motion of the Body.

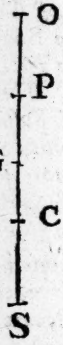
It alſo appears *that the Diſtance of the Center of Gravity from that of Oſcillation* (if the Plane of the Body's Motion remains unalter'd) *will be reciprocally as the Diſtance of the former from the Point of Suſpenſion. Therefore, if that Diſtance be found when the Point of Suſpenſion is in the Vertex, or ſo poſited, that the Operation may become the moſt ſimple, the Value thereof in any other propoſed Poſition of that Point will likewiſe be given, by one ſingle Proportion.*

185. But now, to ſhew how theſe Concluſions may be reduced to Practice, we muſt firſt of all obſerve, that *the Product of any Particle of the Body by the Square of its Diſtance from the Axis of Motion is* (here) *called the Force thereof* (its Efficacy to turn the Body about the ſaid Axis being in that Ratio.) According to which, and the firſt general Value of OC, it appears that, *if the Sum of all the Forces be divided by the Product of the Body into the Diſtance of the Center of Gravity from the Point of Suſpenſion, the Quotient thence ariſing will give the Diſtance of the Center of Percuſſion, or Oſcillation from the ſaid Point of Suſpenſion.*

The Manner of computing the Diviſor has been already explained; it remains therefore to ſhew how the Sum of all the Forces in the Numerator may be collected: Which will admit of ſeveral Caſes. Wherein, to avoid a Multiplicity of Words, I ſhall always expreſs the Diſtance of the Center of Gravity from the Point of Suſpenſion by g, and the Diſtance of the Center of Percuſſion, or Oſcillation, from the ſame Point, by C.

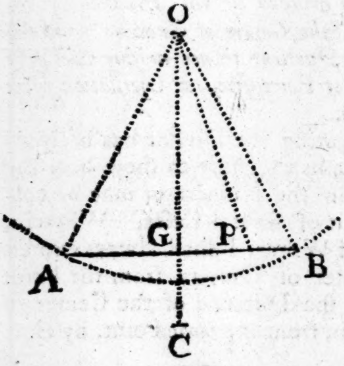
186. Caſe 1. *Let OS be a Line ſuſpended at one of its Extremes.*

Then, if the Part OP (conſidered as variable) be denoted by x, the Force of $\dot{x}$ Particles, at P, will *(as above)* be defined by $\dot{x} \times x^2$: Whoſe Fluent ($\frac{1}{3}x^3$) therefore expreſſes the Force of all the Particles in OP (or the Sum of all the Products, under each Particle, and the Square of its Diſtance from O the Point of Suſpenſion. This Quantity therefore (when x becomes



comes = OS) being divided by OS × $\frac{1}{2}$OS (according to the foregoing Rule or Obſervation) we get $\left(\frac{{}^{1}\mathrm{OS}^3}{\frac{1}{2}\mathrm{OS}^2}=\right)\frac{2}{3}$ OS for the Value of *C*, the true Diſtance of the Center of Oſcillation (or Percuſſion) from the Point of Suſpenſion.

187. Caſe 2. *Let* AB *be a Line, vibrating in a vertical Plane, having its two Extremes* A *and* B *equally diſtant from the Point of Suſpenſion* O.

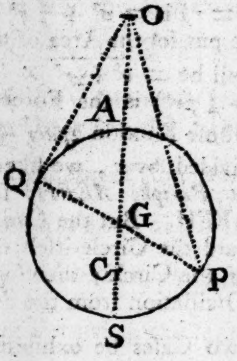


If OG (perpendicular to AB) be put $=a$, and GP $=x$, the Force of $\dot{x}$ Particles at P, will be denoted by $\dot{x}\times\overline{a^2+x^2}=\dot{x}\times$ OP^2*: Whoſe Fluent, divided by ax (or PG × OG) gives $\left(\frac{a^2x+\frac{1}{3}x^3}{ax}\right)a+\frac{x^2}{3a}=\mathrm{OG}+\frac{\mathrm{BG}^2}{3\mathrm{OG}}=C$, when x becomes = GB.

188. Caſe 3. *Let* APSQ *be a Circle, vibrating in a vertical Plane.* Let PQ be any Diameter thereof; then OP^2 + OQ^2 being = 2OG^2 + 2PG^2, the Sum of the Forces of two Particles at P and Q, (putting OG $=a$, and AG $=r$) will be $=\overline{a^2+r^2}\times 2$; whence it is evident that the Sum of the Forces of all the Particles, in the whole Periphery, will be expreſſed by *their Number* $\times\overline{a^2+r^2}$, or by $\overline{a^2+r^2}\times$ *Periph.* APSQ: Which, if

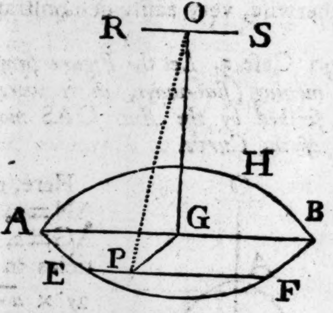
* Art. 185.

if p be put $= 3.141$ *&c.* will be $= \overline{a^2+r^2} \times 2pr = 2pa^2r + 2pr^3$. Hence the Force of the Circle itſelf is alſo given, being $=$ Fluent of $\overline{2pa^2r+2pr^3} \times \dot{r} = pa^2r^2 + \frac{1}{2}pr^4 = \overline{a^2+\frac{1}{2}r^2} \times$ *Circle* APSQ. Now, if the two Expreſſions thus found be divided by $a \times$ *Periph.* APSQ, and $a \times$ *Circle* APSQ reſpectively *, we ſhall have $a + \frac{r^2}{a}$ and $a + \frac{r^2}{2a}$, for the two correſponding Values of C.



189. Caſe 4. *Let* AHBE *be a Circle having its Plane (always) perpendicular to the Axis of Suſpenſion* OG.

Let AGB be that Diameter of the Circle which is parallel to the Axis of Motion RS; and let EF be any Chord parallel to AB and RS; whoſe Diſtance, GP, from the Center of the Circle, let be denoted by x; putting OG $= a$, and AG $= r$: Then, by the Nature of the Circle, EF $= 2\sqrt{r^2-x^2}$; whoſe Diſtance OP, from the Axis of Motion RS, is alſo given $= \sqrt{a^2+x^2}$. Hence it appears that the Force of all the Particles in the Line EF (*defined in Art.* 185.) will be repreſented by $\overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$. Therefore $\dot{x} \times \overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$ is the Fluxion of the Force of the Plane ABFE; whoſe Fluent (when $x=r$)



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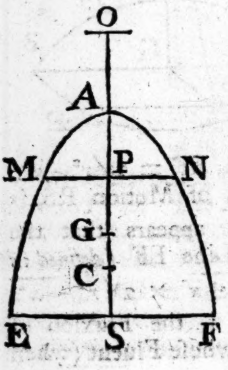
* Art. 185.

$x = r$) is $= \overline{a^2 + \frac{1}{4} r^2} \times$ *Area AEFBG*; which, if p be put for the Area of the Circle whoſe Radius is Unity, will be $= \overline{a^2 + \frac{1}{4} r^2} \times \frac{1}{2} pr^2$; whereof the Double ($pa^2r^2 + \frac{1}{4} pr^4$) is the Force of the whole Circle AEFH: whoſe Fluxion $2par\dot{r} + pr^3\dot{r}$ (ſuppoſing r variable) being divided by $\dot{r}$, we likewiſe get $2pa^2r + pr^3$ ($= \overline{a^2 + \frac{1}{2} r^2} \times$ *Periph. AEFH*) for the Force of the Periphery AEFH. But the Center of Gravity, whether we regard the Circle itſelf or its Periphery, is in the Center of the Circle; therefore the Diſtance of the Center of Oſcillation from the Point of Suſpenſion, will in theſe two Caſes be exhibited by $a + \frac{r^2}{4a}$ and $a + \frac{r^2}{2a}$ reſpectively.

If the Circle, inſtead of being perpendicular to GO, either coincides, or makes a given Angle with it, the Value of C will come out exactly the ſame; provided the Diameter AB ſtill continues parallel to the Axis of Motion RS: This appears from *Art.* 184. and may be, otherwiſe, very eaſily demonſtrated.

190. Caſe 5. *Let the Figure propoſed be a Curve* AEF, *moving (flat-ways, as it were) ſo that the Plane deſcribed by the Axis* OAS *may be perpendicular to that of the Curve.*



Here, putting AP $= x$, PN $= y$, AN $= z$, OA $= d$, OG $= g$, and AG $= a$, the Force of the Particles in MN will be defined by $2y \times \overline{a + x}|^2$. Therefore the Fluent of $2y\dot{x} \times \overline{d + x}|^2$ will be as the whole Force of the Plane NAM (or AEF, when $x =$ AS) and conſequently $C = \frac{Flu.\ \overline{d + x}|^2 \times y\dot{x}}{Flu.\ \overline{d + x} \times y\dot{x}}$: Which, therefore,

fore, when the Point of Suſpenſion is in the Vertex A, will become $C = \frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}$. Let this Value be denoted by v; then, the Diſtance of the Centers of Gravity and Oſcillation being $v - a$, we have (by *Art.* 184.) $g : a :: v - a : \left(\frac{a \times \overline{v-a}}{g}\right)$ the Diſtance of the ſame Centers, when the Point of Suſpenſion is at O, and conſequently C, in that Caſe, $= g + \frac{a \times \overline{v-a}}{g}$: Which Form will be found more commodious than the foregoing in moſt Caſes.

After the ſame Manner the Value of C, with reſpect of the Arch AEF, will appear to be $= \frac{Flu.\ \overline{d+x}|^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$ $= g + \frac{a \times \overline{v-a}}{g}$, ſuppoſing $v = \frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}$.

It may not be improper to give an Example or two of the Uſe of the foregoing Theorems.

191. Let therefore EAF be, firſt, conſider'd as an iſoſceles Triangle: In which Caſe AP (x) and PN (y) being in a conſtant Ratio, we have $y = \frac{bx}{c}$ (ſuppoſing SF$=b$ and AS$=c$.) Hence $C \left(= \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}\right)$ $= \frac{Flu.\ d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x}}{Flu.\ dx\dot{x} + x^2\dot{x}} = \frac{\frac{1}{2}d^2 + \frac{2}{3}dx + \frac{1}{4}x^2}{\frac{1}{2}d + \frac{1}{3}x} =$ $\frac{6d^2 + 8dx + 3x^2}{6d + 4x}$: Or (according to the ſecond Form) becauſe $v \left(\frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right) = \frac{3x}{4}$, and a is known to

be

* Art. 175. be $= \frac{2x}{3}$*, we have $C\left(=g+\frac{a\times\overline{v-a}}{g}\right)=g+\frac{x^2}{18g}$, where $g\ (=d+a)=d+\frac{2}{3}x$.

Again, becauſe $\dot{z}$ and $\dot{x}$ are in a conſtant Ratio, we alſo have $\frac{Flu.\ \overline{d+x}|^2\times\dot{z}}{Flu.\ \overline{d+x}\times\dot{z}}=\frac{Flu.\ \overline{d+x}|^2\times\dot{x}}{Flu.\ \overline{d+x}\times\dot{x}}=\frac{d^2+dx+\frac{1}{3}x^2}{d+\frac{1}{2}x}$; whence the Center of Oſcillation of the Lines EH and AF is given.

192. For a ſecond Example, let EAF be ſuppoſed a Parabola of any Kind, whoſe Equation is $y=\frac{x^n}{c^{n-1}}$: Then (*according to Form* 2.) we ſhall firſt have $v\left(=\frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right)=\frac{Flu.\ x^{n+2}\dot{x}}{Flu.\ x^{n+1}\dot{x}}=\frac{\overline{n+2}\times x}{n+3}$: Whence,

† Art. 176. a being $=\frac{\overline{n+1}\times x}{n+2}$†, we alſo get $C\left(=g+\frac{a\times\overline{v-a}}{g}\right)=g+\frac{\overline{n+1}\times x^2}{\overline{n+2}|^2\times\overline{n+3}\times g}$; where $g=d+\frac{\overline{n+1}\times x}{n+2}$.

But, with reſpect to the Arch of the Curve, $v\left(=\frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}\right)$ is $=\frac{Flu.\ x^2\dot{x}\sqrt{c^{2-2}+nnx^{2n-2}}}{Flu.\ x\dot{x}\sqrt{c^{2n-2}+nnx^{2n-2}}}$: From which Value (found by infinite Series, and even with-

‡ Art. 138. out in ſome Caſes ‡) that of C will alſo be given.

193. Caſe. 6. *Let the propoſed Figure be a Curve vibrating (edge-ways) ſo that the Motion of the Axis may be in the Plane of the Curve.*

Then (*by Caſe* 2.) the Force of all the Particles in the Line PN (*ſee the preceding Figure*) being defined by $OP^2\times PN+\frac{1}{3}PN^3$, or $\overline{d+x}|^2\times y+\frac{1}{3}y^3$ (retaining the No-

Notation above) we have $C = \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, when the Point of Suſpenſion is in the Vertex A, will become $\frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}$: Let this (when found) be denoted by v; then, it appears from the preceding Caſe, that the general Value of C will, *alſo*, be repreſented by $g + \frac{a \times \overline{v - a}}{g}$.

In the ſame manner the Value of C, with reſpect to the Arch EAF, will be expounded by $\frac{Flu.\ \overline{\overline{d+x}|^2 + y^2} \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$, or by $g + \frac{a \times \overline{v - a}}{g}$, ſuppoſing $v = \frac{Flu.\ \overline{x^2 + y^2} \times \dot{z}}{Flu.\ x\dot{z}}$.

194. Example. *Let the Equation of the given Curve be* $y = \frac{x^n}{c^{n-1}}$: Then $v \left(= \frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}} \right) =$

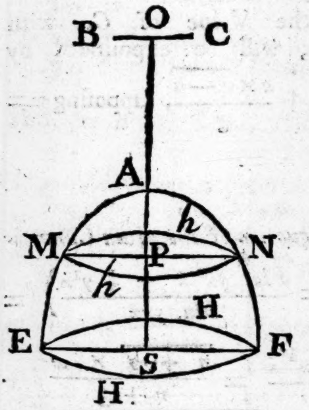
$$\frac{Flu.\ c^{1-n}x^{n+2}\dot{x} + \frac{1}{3}c^{3-3n}x^{3n}\dot{x}}{Flu.\ c^{1-n}x^{m+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3} +$$

$$\frac{\frac{1}{3}c^{2-2n}x^{3n+1} \times \overline{n+2}}{\overline{3n+1} \times x^{n+2}} = \frac{\overline{n+2} \times x}{x+3} + \frac{\overline{n+2} \times c^{2-2n} \times x^{2n-1}}{3 \times \overline{3n+1}}$$

$= \frac{n+2}{n+3} \times x + \frac{\overline{n+2}}{3 \times \overline{3n+1}} \times \frac{y^2}{x}$: From which the Value of C is alſo given; and from whence it appears, that if n be expounded by 0, v will become $= \frac{2x}{3} + \frac{2y^2}{3x} = \frac{2}{3} \times \frac{x^2 + y^2}{y}$; in which Caſe the Figure will degenerate to a Rectangle: But, if n be interpreted by 1, the Figure EAF will then be an iſoſceles

Triangle, and $v = \frac{3x}{4} + \frac{y^2}{4x}$: Laſtly, if n be taken $= \frac{1}{2}$, the Curve will be the common Parabola, and $v = \frac{5x}{7} + \frac{c}{3}$.

195. *Caſe* 7. *Let the Figure* AEFH *be a Solid generated by the Rotation of a Curve* EAF *about its Axis* AS ; *having its Baſe* HH *parallel to the Axis of Motion* BOC.



It appears, *from Caſe* 4. that the Force of all the Particles in the circular Section *hh* (parallel to HH) will be expreſſed by $\overline{OP^2 + \frac{1}{4} PN^2} \times$ *Circle hh*, or $\overline{OP^2 \times PN^2 + \frac{1}{4} PN^4} \times p$ (p being $= 3.1415$ *&c.*) which, in algebraic Terms, is $\overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4} y^4} \times p$. Hence we have $C =$

$$\frac{Flu.\ \overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4} y^4} \times p\dot{x}}{Flu.\ \overline{d+x} \times py^2\dot{x}}\,{}^{*} = \frac{Flu\ \overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4} y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}.$$

* Art. 185.

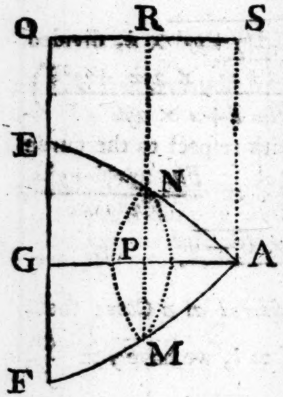
Which, therefore, when the Point of Suſpenſion is in the Vertex A, becomes $\frac{Flu.\ y^2x^2\dot{x} + \frac{1}{4} y^4\dot{x}}{Flu.\ y^2x\dot{x}} = v$; and conſequently $C = g + \frac{a \times \overline{v - a}}{g}$, as in the preceding Caſes.

But, with regard to the Superficies of the Solid ; it is found, *in Caſe* 4. that the Force of the Particles in the Periphery M*h*N*h* is expreſſed by $\overline{OP^2 + \frac{1}{2} PN^2} \times$ *Periph. Mh Nh* $= \overline{d+x}|^2 \times 2py + py^3$.

Hence the Fluent of $\overline{\overline{d+x}|^{2} \times 2py+py^{3}} \times \dot{z}$, divided by that of $\overline{d+x} \times 2py\dot{z}$ $\left(= \frac{Flu.\ \overline{d+x}|^{2} \times 2y\dot{z} + y^{3}\dot{z}}{Flu.\ d+x \times 2y\dot{z}}\right)$ will give the true Value of C with reſpect to the curve Surface E*b*A*b*F. Which, putting $v = \frac{Flu.\ 2yx^{2}\dot{z}+y^{3}\dot{z}}{Flu.\ 2yx\dot{z}}$, is likewiſe expreſſed by $g + \frac{a \times \overline{v-a}}{g}$.

196. Ex. 1. *Let* EAF *be conſidered as a Cone*; then, putting AS $=f$, SF $=b$ and AF $=c$, we have $y = \frac{bx}{f}$, $z = \frac{cx}{f}$; and therefore C $\left(= \frac{Flu.\ \overline{d+x}|^{2} \times y^{2}\dot{x} + \frac{1}{4}y^{4}\dot{x}}{Flu.\ \overline{d+x} \times y^{2}\dot{x}}\right)$ $= \frac{20d^{2}+30fd+12f^{2}+3b^{2}}{20d+15f}$, when $x = f$. But, with reſpect to the convex Superficies, C will be found $= \frac{12d^{2}+16df+6f^{2}+3b^{2}}{12d+8f}$.

197. Ex. 2. *Let* EAF *&c. be conſidered as a Sphere whoſe Center is* S, *and Radius* AS$=r$; in which Caſe, y^{2} being $= 2rx-x^{2}$, we have v $\left(= \frac{Flu.\ y^{2}x^{2}\dot{x} + \frac{1}{4}y^{4}\dot{x}}{Flu.\ y^{2}x\dot{x}}\right)$ $= \frac{Flu.\ r^{2}x^{2}\dot{x}+rx^{3}\dot{x}-\frac{3}{4}x^{4}\dot{x}}{Flu.\ 2rx^{2}\dot{x}-x^{3}\dot{x}} = \frac{\frac{1}{3}r^{2}+\frac{1}{4}rx-\frac{3}{20}x^{2}}{\frac{2}{3}r-\frac{1}{4}x}$; whence C is alſo given. But, when $x = 2r$ (or the whole Sphere is taken) $v = \frac{7r}{5}$: Therefore a being$=r$, and g = OS, in this Caſe, we have C $\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{r \times 2r}{5g} = g + \frac{2r^{2}}{5g}$.



198. Caſe 8. *Let the Figure propoſed be a Solid, as in the preceding Caſe, but let its Axis* AG *be, here, parallel to the Axis of Motion* ORS.

Then, if RP (OG) be put $= g$, 3,1459 &c. $= p$, AP $= x$ &c. the Force of the Particles in the Circle NM (parallel to EF) will be exhibited by $\overline{g^2 + \frac{1}{2}y^2} \times py^2$, or $pg^2y^2 + \frac{1}{2}py^4$ (*Vid. Caſe* 3.) Hence we have $C =$

* Art. 185. † Art. 145. $\frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Solid}$ * $= \frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Flu.\ py^2\dot{x}}$ ‡ $=$ $g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}$.

Moreover, with reſpect to the Superficies; the Force of the Particles in the Periphery of the ſaid Circle MN

‡ Art. 185. being $\overline{2pg^2y + 2py^3}$ ‡, we have, in this Caſe, $C =$ $\frac{Flu.\ \overline{2pg^2y + 2py^3} \times \dot{z}}{g \times Superficies.} = \frac{Flu.\ 2pg^2y\dot{z} + 2py^3\dot{z}}{g \times Flu.\ 2py\dot{z}} = g +$ $\frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}}$.

199. Ex. 1. *Let* EAF *be a Segment of a Sphere, whoſe Radius is r*; then y^2 being $= 2rx - x^2$, we ſhall have

$$C\ \left(g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}\right) = g + \frac{Flu.\ 2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x}}{g \times Flu.\ \overline{2rx\dot{x} - x^2\dot{x}}}$$

$$= g + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{g \times \overline{r - \frac{1}{3}x}} = g + \frac{\overline{20r^2 - 15rx + 3x^2} \times x}{\overline{30r - 10x} \times g}.$$

Which, when x is expounded, either, by r or $2r$, becomes $= g + \frac{2r^2}{5g}$, for the true Value of C, when either the Hemiſphere, or whole Sphere, is taken. But, with reſpect to the Center of Oſcillation of the Superficies thereof, we have $\dot{z}$ in this Caſe $= \frac{r\dot{x}}{\sqrt{2rx - xx}}$[*] $= \frac{r\dot{x}}{y}$: And therefore $g + \frac{Flu.\, y^3\dot{z}}{g \times Flu.\, y\dot{z}} = g + \frac{Flu.\, \overline{2rx - xx} \times r\dot{x}}{g \times Flu.\, r\dot{x}} = g + \frac{rx - \frac{1}{3}x^2}{g}$: Whieh, when $x = r$, or $x = 2r$, becomes $g + \frac{2r^2}{3g}$.

200. Ex. 2. *Let the Solid* EAF *be a Paraboloid, whoſe generating Curve is defined by the Equation* $y = \frac{x^n}{c^{n-1}}$:

Then $C = g + \frac{Flu.\, \frac{1}{2}y^4\dot{x}}{g \times Flu.\, y^2\dot{x}} = g + \frac{Flu.\, \frac{1}{2}x^{4n}\dot{x} \times c^{4-4n}}{g \times Flu.\, x^{2n}\dot{x} \times c^{2-2n}}$ $= g + \frac{\overline{2n+1} \times x^{2n}}{\overline{4n+1} \times 2g \times c^{2n-2}} = g + \frac{\overline{2n+1} \times y^2}{\overline{4n+1} \times 2g}$. Where, if n be taken $= 0$, the Figure will become a Cylinder, and $C = g + \frac{y^2}{2g}$: But if n be expounded by 1, the Figure will be a Cone, and $C = g + \frac{3y^2}{10g}$. Laſtly, if n be taken $= \frac{1}{2}$, the Figure will be the Solid generated from the common Parabola and $C = g + \frac{y^2}{3g}$.

[*] Art. 142.

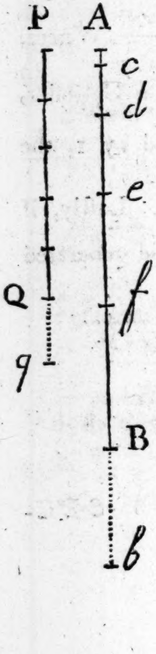
SECTION XII.

Of the Uſe of FLUXIONS *in determining the Motion of Bodies affected by centripetal Forces.*

PROPOSITION I.

201. THE *Motion, or Velocity, acquired by a Body freely deſcending from Reſt, by the Force of an uniform Gravity, is proportional to the Time of its Deſcent; and the Space gone over, as the Square of that Time.*

The firſt Part of the Propoſition is almoſt ſelf-evident: For, ſince any Motion is proportional to the Force by which it is generated, that generated by the Force of an uniform Gravity muſt be as the Time of Deſcent; becauſe the whole Effect of ſuch a Force, acting equally every Inſtant, is as that Time.



Let, now, the Velocity acquired during a Deſcent of one Second of Time, be ſuch as would carry the Body uniformly over any Diſtance b in one Second; and let AB (x) denote the Diſtance deſcended in any propoſed Time t; which Time let be denoted by PQ; making $Bb = \dot{x}$ and $Qq = \dot{t}$: Then it will be, as $1'' : t :: b : (bt)$ the Diſtance that would be uniformly deſcribed in $1''$, with the Velocity at B: Alſo $1'' : \dot{t} ::$ the ſaid Diſtance (bt) to $bt\dot{t} = \dot{x}$ *. By taking the Fluent whereof we get

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* Art. 2.

$\frac{1}{2}bt^2 = x$. Therefore the Diſtance deſcended ($\frac{1}{2}bt^2$) is as the Square of the Time. *Q. E. D.*

Otherwiſe, without Fluxions.

Conceive the Time, (PQ) of falling thro' AB to be divided into an indefinite Number of very ſmall equal Particles, repreſented, each, by m; and let the Diſtance deſcended in the firſt of them be Ac, in the ſecond cd, in the third de, *&c. &c.* Then, the Velocity being always as the Time from the Beginning of the Deſcent, it will in the Middle of the firſt of the ſaid Particles be defined by $\frac{1}{2}m$; in the Middle of the ſecond by $1\frac{1}{2}m$; in the Middle of the third by $2\frac{1}{2}m$, *&c. &c.* But, ſince the Velocity at the Middle of any Particle of Time, is a Mean between thoſe at the two Extremes, or betwixt any other two equally remote from it, the correſponding Particle of the Diſtance AB may, therefore, be conſidered as deſcribed by that mean Velocity. And ſo, the Spaces Ac, cd, de, ef, *&c.* deſcribed in equal Times, being reſpectively as the ſaid mean Celerities $\frac{1}{2}m$, $1\frac{1}{2}m$, $2\frac{1}{2}m$, $3\frac{1}{2}m$, *&c.* it follows, by Addition, that the Diſtances, Ac, Ad, Ae, Af, *&c.* gone over from the Beginning, are to one another as $\frac{m}{2}$, $\frac{4m}{2}$, $\frac{9m}{2}$, $\frac{16m}{2}$, *&c.* or 1, 4, 9, 16, 25, *&c.* that is, as the Squares of the Times. *Q. E. D.*

COROLLARY I.

202. Since the Diſtance that might be uniformly run over in one Second, with the Velocity at B, is expreſſed by bt, the Diſtance that might be deſcribed with the ſame Velocity in the Time t will therefore be expreſſed by $bt \times t$, or bt^2: Whence it appears, that the Space AB ($\frac{1}{2}bt^2$) thro' which the Body falls in any given Time t, is juſt the half of that which would be uniformly deſcribed with the Celerity at B, in the ſame Time.

Therefore, ſince it is found from Experiment, that a Body near the Earth's Surface (where the Gravity may be taken as uniform) deſcends about $16\frac{1}{12}$ Feet in the firſt Second, it follows that the Value of b (is in this Caſe) $= 2 \times 16\frac{1}{12} = 32\frac{1}{6}$: And conſequently the Number of Feet deſcended in t Seconds, equal to $16\frac{1}{12} \times t^2$.

COROLLARY 2.

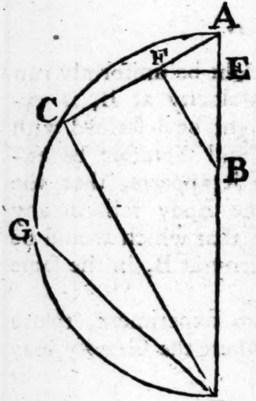
203. It is evident, whatever Force the Body deſcends by, the Value of b will always be as that Force; ſince a double Force, in the ſame time, generates a double Velocity; a treble Force, a treble Velocity, *&c.* Therefore, ſeeing our Equation $\frac{1}{2}bt^2 = x$, alſo gives $t = \sqrt{\frac{x}{\frac{1}{2}b}}$, and $b = \frac{x}{\frac{1}{2}t^2}$, it follows,

1. That the Diſtance deſcended is, univerſally, as the Force and the Square of the Time conjunctly.

2. That the Time is always as the Square-root of the Diſtance applied to the Force.

3. And that the Force is as the Diſtance apply'd to the Square of the Time —— What is above demonſtrated with reſpect to the Times, holds alſo in the Velocities, when the accelerating Forces are equal.

PROPOSITION II.



204. *To determine the Velocity, and Time of Deſcent, of a Body along an inclined Plane* AC.

From any Point F, in AC, draw FE perpendicular to the vertical Line AD, and make FB and CD perpendicular to AC, meeting AD in B and D. Becauſe (by the Principles of Mechanics) the Force of Gravity in the Direction CF, whereby the Body is made to deſcend along the Plane, is to the abſolute Force thereof, as AF to

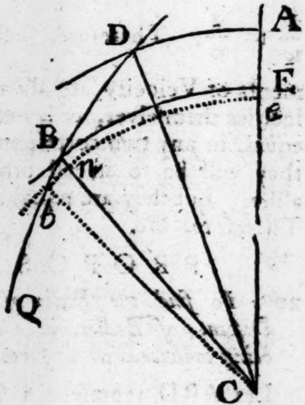
AB, or as AC to AD; and ſince (*by Caſe* 1. *Art.* 203.) the Diſtances deſcended in equal Times are as the Forces, it follows that the Time of Deſcent thro' AF will be equal to the Time of the perpendicular Deſcent thro' AB: And conſequently the Time of Deſcent thro' AC equal to that thro' AD; which is given by *Prop.* 1. Moreover, becauſe the Velocities at F and B, acquired in equal Times, are as the Forces, or as AF to AB; and it appears from *Prop.* 1. that the Velocity at E is to that at B, as $\sqrt{AE} : \sqrt{AB}$, or as $\sqrt{AE \times AB}$ $(=AF) : \sqrt{AB \times AB}$ $(= AB)$ it follows, by Equality, that the Celerity at F is equal to that at E; which is therefore given, by the preceding Propoſition. *Q. E. I.*

COROLLARY.

205. Hence the Time of Deſcent along the Chord AC of a Semi-circle ACD is equal to the Time of Deſcent along the vertical Diameter AD: And, if the Chord DG be of the ſame Length with AC (its Inclination to the Horizon being alſo the ſame) the Time of Deſcent along it will alſo be equal to that along the vertical Diameter.

PROPOSITION III.

206. *If, from two Points* A *and* D, *equally remote from the Center of Attraction* C, *two Bodies move, with equal Celerities, the one along the Right-line* AC, *the other in a Curve-line* DBQ, *their Celerities at all other equal Diſtances from the Center, will be equal.*



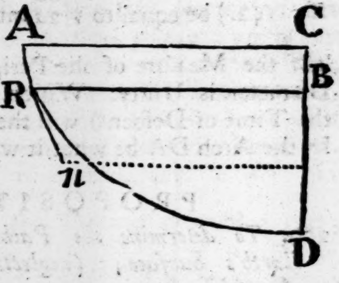
For, let CB and CE be any two ſuch Diſtances; let the Arch BE be deſcribed, from the Center C, and alſo *eb*, indefinitely near to it, cutting CB in *n*: Let the centripetal Force at the Diſtance of CB, or CE, be repreſented by *f*, and the Velocity at B, by *v*.

By the Reſolution of Forces, the Efficacy of the Force (f) in the Direction B*b*, whereby the Velocity of the Body is accelerated, will be $\frac{Bn}{Bb} \times f$: Alſo the Time of moving over B*b* (being as the Diſtance apply'd to the Velocity) is repreſented by $\frac{Bb}{v}$: Therefore the Increaſe of Velocity, in moving thro' B*b*, being as the Force and Time conjunctly, will be defined by $\frac{Bn}{Bb} \times f \times \frac{Bb}{v}$, or its Equal $\frac{Bn}{v} \times f$. In the ſame Manner, the Velocity at E being denoted by *w*, the Time of falling thro' E*e* will be repreſented by $\frac{Ee}{w}$, and the Velocity generated in that Time by $\frac{Ee}{w} \times f$: Which is to that $\left(\frac{Bn}{v} \times f\right)$ acquired in falling thro' the Arch B*b*, as $\frac{1}{w}$ to $\frac{1}{v}$. Therefore, ſeeing the correſponding Increments of Velocity are always, reciprocally, as the Velocities themſelves, it is manifeſt, if thoſe Velocities are equal, in any two correſponding Poſitions of the Bodies, they will be ſo in all others, being always increaſed alike. But they are equal at A and D by Suppoſition: Therefore, &c. *Q. E. D.*

PROPOSITION IV.

207. *To find the Ratio of the Velocities, and Times of Deſcent, of Bodies, in Curves; the Force of Gravity being conſidered as uniform.*

Let ARD repreſent a Curve of any Kind, along which a Body deſcends, by the Force of its own Gravity from A; let AC, RB, &c. be parallel, and CD perpendicular, to the Horizon; moreover, let Rn touch the Curve at R; and let CB $= u$, AR $= w$, and R$n = \dot{w}$ [*].



Since the Points B and R (as well as C and A) may be looked upon as equally remote from the Earth's Center (to which the Gravitation tends), the Velocity acquired in deſcending thro' the Arch AR will (*by the laſt Propoſition*) be equal to that acquired by falling freely through the Right-line CB; which laſt Velocity (*by Prop.* 1.) is always as $\sqrt{CB}$ (or $u^{\frac{1}{2}}$). Therefore the Celerity, whether the Body moves in a Right-line, or a Curve, is always in the ſubduplicate Ratio of the perpendicular Deſcent; and ſo, the Time in which Rn ($\dot{w}$) would be uniformly deſcribed, with that Celerity, will be univerſally as $\frac{\dot{w}}{u^{\frac{1}{2}}}$; whoſe Fluent is as the Time of falling through AR. *Q. E. I.*

EXAMPLE.

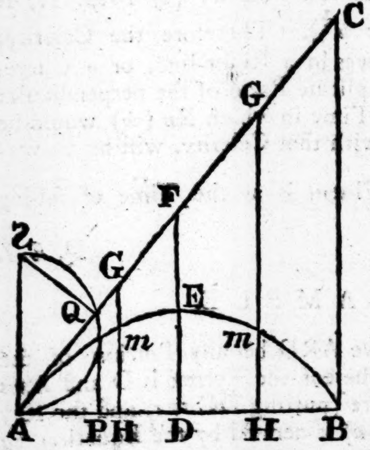
208. Let the Curve ARD be any Portion of the common Cycloid; whereof the Vertex is D and Axis DC; and whoſe Nature (putting DC $= c$, and the Ray of Curvature at D $= a$) is defined by the Equation $2a \times DB = DR^2$. Here, we have DR ($= \sqrt{2a} \times \sqrt{DB}$) $= \sqrt{2a} \times \overline{c - u}|^{\frac{1}{2}}$; whoſe Fluxion $- \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\overline{c - u}|^{\frac{1}{2}}}$, with a contrary Sign, is the Value of Rn or $\dot{w}$; and

[*] Art. 135.

therefore $\frac{\dot{w}}{u^{\frac{1}{2}}} = \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\sqrt{cu - uu}}$: Whoſe Fluent, at the loweſt Point D, where u becomes $= c$, will (*by Art.* 142.) be equal to $\sqrt{2a}$ multiplied by $\left(\frac{3.14159\&c.}{2}\right)$ half the Meaſure of the Periphery of the Circle whoſe Diameter is Unity. Which Fluent (and conſequently the Time of Deſcent) will therefore continue the ſame, let the Arch DA be what it will.

PROPOSITION V.

209. *To determine the Paths of Projectiles near the Earth's Surface; (neglecting the Reſiſtance of the Atmoſphere.)*



Let a Body be projected from the Point A, in the Direction of the Line AC, with a Velocity ſufficient to carry it uniformly over the Diſtance d in the Time t; and let the Space through which it would freely deſcend, by its own Gravity, in that time, be denoted by b; alſo let the Sine of the Angle of Elevation BAC (to the Radius r) be put $= s$, its Co-ſine $= c$, and the Diſtance of the Point A from the Ordinate Hm (conſidered as moving parallel to itſelf along with the Body) $= x$; then, *by Trig.* HG (perpendicular to AB) will be $= \frac{sx}{c}$, and AG $= \frac{rx}{c}$.

Becauſe the Projectile is turned aſide, continually, from a rectilinear Path, by the Earth's Attraction, it muſt deſcribe a Curve-line $AmEmB$, to which AC is a Tangent at the Point A: But that Attraction, acting always in a Direction (Hm) perpendicular to the Horizon, can have no Effect upon that Part of the Velocity with which the Body approaches the Line BC, parallel to Hm; therefore the Right-line HG (in which the Body is always found) will continue to move uniformly towards BC, the ſame as if Gravity was not to act; and the Diſtance Gm deſcended from the Tangent AC, by means of the Attraction, will be the very ſame as if the Body was to deſcend from Reſt along the Line GH. This being premiſed, it is evident, that as d : AG $\left(\frac{rx}{c}\right) :: t : \left(\frac{rx}{cd} \times t\right)$ the Time of deſcribing Am; and, as $t^2 : \frac{r^2x^2}{c^2d^2} \times t^2 :: b : \left(\frac{br^2x^2}{c^2d^2}\right)$ the Space (Gm) through which a Body would freely deſcend in that Time (*by Prop.* 1.)

Hence $\frac{sx}{c} - \frac{br^2x^2}{c^2d^2}$, or $\frac{csd^2x - br^2x^2}{c^2d^2}$ is a general Value for the Ordinate Hm: By putting which $= 0$, we get $x = \frac{csd^2}{br^2} =$ AB $=$ the Amplitude of the Projection. But, by putting its Fluxion equal to nothing, we have $x = \frac{csd^2}{2br^2}$; which ſubſtituted for x in the Value of Hm, gives $\frac{s^2d^2}{4br^2}$ for the Altitude DE of the Projection. *Q.E.I.*

COROLLARY.

210. If another Body be projected, with the ſame Celerity, in the vertical Direction AS; then, s becoming $= r$, the Altitude of that Projection $\left(\frac{s^2d^2}{4br^2}\right)$ will be-

come $\frac{d^2}{4b} =$ AS; which call h, and let this Value be ſubſtituted in thoſe of AB and DE, and they will become $\frac{4hcs}{r^2}$ and $\frac{hs^2}{r^2}$ reſpectively.

Hence, if from the Point Q where the Line of Direction AC cuts a Semi-circle deſcribed upon AS, the Lines SQ and QP be drawn, the latter perpendicular to AB, the Triangles ASQ and AQP being ſimilar, we ſhall have

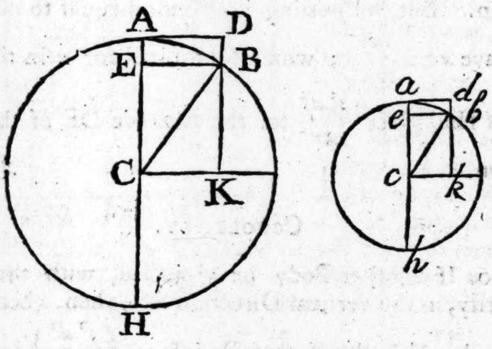
$$r : s :: h\ (\mathrm{AS}) : \frac{sh}{r} = \mathrm{AQ}$$

$$r : s :: \frac{sh}{r}\ (\mathrm{AQ}) : \frac{s^2h}{r^2} = \mathrm{PQ} = \mathrm{DE}$$

$$r : c :: \frac{sh}{r^2}\ (\mathrm{AQ}) : \frac{sch}{r} = \mathrm{AP} = \tfrac{1}{4}\,\mathrm{AB}$$

PROPOSITION VI.

211. *To determine the Ratio of the Forces, whereby Bodies, tending to the Centers of given Circles, are made to revolve in the Peripheries thereof.*



Let ABH and *abh* be any two propoſed Circles, whereof let AB and *ab* be ſimilar Arcs; in which, let the Velocities of the revolving Bodies be reſpectively as *V* to *v*; make DBK and *dbk* parallel to the Radii AC and *ac*, putting AC $= R$, $ac = r$, and the Ratio of the centripetal Force in ABH to that in *abh*, as *F* to *f*.

It is plain, becauſe the Angles ABD and *abd* are equal, that the Velocities at B and *b*, in the Directions BK and *bk*, with which the Bodies recede from the Tangents AD and *ad*, are to each other as the abſolute Celerities *V* and *v* *. But thoſe Velocities, being the Effects of the centripetal Forces acting in correſponding, ſimilar, Directions during the Times of deſcribing AB and *ab*, will therefore be as the Forces themſelves when the Times are equal; but when unequal, as the Forces and Times conjunctly. Therefore, the Times being univerſally as $\frac{AB}{V}$ to $\frac{ab}{v}$, or as $\frac{R}{V}$ to $\frac{r}{v}$ (becauſe the Arcs AB and *ab* are ſimilar) we have, as $F \times \frac{R}{V} : f \times \frac{r}{v} :: V : v$; whence (multiplying the Antecedents by $\frac{V}{R}$ and the Conſequents by $\frac{v}{r}$) it will be, as $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$: Therefore the Forces are as the Squares of the Velocities directly, and as the Radii inverſely.

Otherwiſe.

Let the indefinitely little Arch AB be the Diſtance that the Body moves over in a given, or conſtant Particle of Time; and let the centripetal Force at B be meaſured by twice the Subtenſe or Space AE through which the Body is drawn from the Tangent AD in that Time †.

Then,

* Art. 35.

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† *The Velocity which any Force, uniformly continued, is capable of generating, in a given Body, in a given Time, is the proper Meaſure of the Intenſity of that Force* *. *But this Velocity is itſelf meaſured by the Space the Body would move uniformly*

* Art. 203.

Then, by the Nature of the Circle, $AB^2 = AH \times AE = AC \times 2AE$, and conſequently $2AE = \frac{AB^2}{AC}$: Therefore, the Force is as the Square of the Velocity applied to the Radius of the Circle *(as before)*.

COROLLARY I.

212. Becauſe, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$

$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

COROLLARY II.

213. If the Ratio of the periodic Times be denoted by that of P to p; then the Ratio of the Velocities V, v being as $\frac{R}{P}$ to $\frac{r}{p}$, we ſhall have, by Equality $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence alſo

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$

$$R : r :: FP^2 : fp^2.$$

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formly over in a given Time; which Space is always the double of that through which the Body would freely deſcend, from Reſt, in the ſame time. Therefore* 2AE *is the proper Meaſure of the centripetal Force, according as we have aſſumed it.——It is true, when the Forces to be compared are all computed in the ſame Manner, from the Naſcent, or indefinitely ſmall Subtenſes of contemporaneous Arcs, it matters not whether we conſider thoſe Subtenſes, or their Doubles, as the Meaſures of the Forces, the Ratio being the ſame in both Caſes. But when the Forces ſo found are to be compared with others derived from a fluxional Calculus, it is abſolutely neceſſary to take the double Subtenſe for the Meaſure of the Force.——This Note is inſerted, that the Learner may avoid the Errors, which ſome very conſiderable Mathematicians have fallen into by not properly attending to this Particular.*

* Art. 202.

Co-

Corollary III.

214. If the Meaſure of the Force, or the Velocity that might be uniformly generated in a given Time (1) be expounded by any Power a^n of the Radius AC (a); then the Diſtance through which a Body would freely deſcend in the ſame Time, by that Force, uniformly continued, will be expreſſed by $\frac{1}{2}a^n$ *. Therefore, the Diſtances deſcended, by means of the ſame Force, uniformly continued, being as the Squares of the Times †, it is evident, if the Time of moving through AB be denoted by t, that the Diſtance AE deſcended in that Time, will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2}a^n$: And ſo we ſhall have AB $(\sqrt{2AE \times AC}) = \frac{t}{1} \times a^{\frac{n+1}{2}}$; which being the Diſtance deſcribed by the revolving Body in the Time t, it follows that the Space gone over in the given Time (1) will be $a^{\frac{n+1}{2}}$: Which, therefore, is the true Meaſure of the Celerity in this Caſe. The ſame concluſion might have been derived in much fewer Words from *Corol.* 1. but, as a thorough underſtanding hereof is abſolutely neceſſary in what follows hereafter, I have endeavoured to make it as plain as poſſible.

* Art. 202.

† Art. 201.

Corollary IV.

215. Hence the Time of Revolution is alſo derived; for it will be as $a^{\frac{n+1}{2}} : 3.14159$ &c. $\times\, 2a$ (the whole Periphery) :: $1 : \frac{3.14 \text{ \&c.} \times 2a}{a^{\frac{n+1}{2}}}$ or 3.14159 &c. $\times\, 2a^{\frac{1-n}{2}}$, the true Meaſure of the periodic Time.

COROLLARY V.

216. Therefore, if n be expounded by 1, 0, -1, -2 and -3 ſucceſſively, then the Velocity correſponding will be as a, $a^{\frac{1}{2}}$, 1, $a^{-\frac{1}{2}}$, and a^{-1}; and the Time of Revolution, as 1, $a^{\frac{1}{2}}$, a, $a^{\frac{3}{2}}$ and a^2 reſpectively.

SCHOLIUM.

217. From the preceding Propoſition, and its ſubſequent Corollaries, the Velocity and periodic Time of a Body revolving in a Circle at any given Diſtance from the Earth's Center, by means of its own Gravity, may be deduced: For let d be put for the Space thro' which a heavy Body, at the Surface of the Earth, deſcends in the firſt Second of Time, then $2d$ will be the Meaſure of the Force of Gravity at the Surface: And therefore, the Radius of the Earth being denoted by r, the Velocity, *per* Second, in a Circle at its Surface, will be $\sqrt{2rd}$; and the Time of Revolution $= \frac{3.14159 \,\&c. \times 2r}{\sqrt{2rd}}$ $= 3.14159 \,\&c. \times \sqrt{\frac{2r}{d}}$ *(Seconds)*; which two Expreſſions, becauſe r is $= 21000000$ Feet and $d = 16\frac{1}{12}$ will therefore be nearly equal to 26000 Feet and 5075 *Seconds*, reſpectively. Let R be now put for the Radius of any other Circle deſcribed by a Projectile about the Earth's Center: Then, becauſe the Force of Gravitation above the Surface is known to vary according to the Square of the Diſtance inverſely, we have *(by Caſe* 4. *Corol.* 5.) $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} :: (26000 \overset{F}{)}$ the Velocity *(per* Second) at the Surface, to $26000 \times \sqrt{\frac{r}{R}}$, the Ve-

locity in the Circle whoſe Radius is *R*: And $r^{\frac{3}{2}} : R^{\frac{3}{2}}$ :: $(\overset{s.}{5075})$ the periodic Time at the Surface : to $5075 \times \sqrt{\frac{R^3}{r^3}}$, the Time of Revolution in the Circle *R*. Which, if *R* be aſſumed equal to $(60r.)$ the Diſtance of the Moon from the Earth, will give $\overset{s.}{2360000}$, or $\overset{D}{27,3}$ nearly, for the periodic Time at that Diſtance.

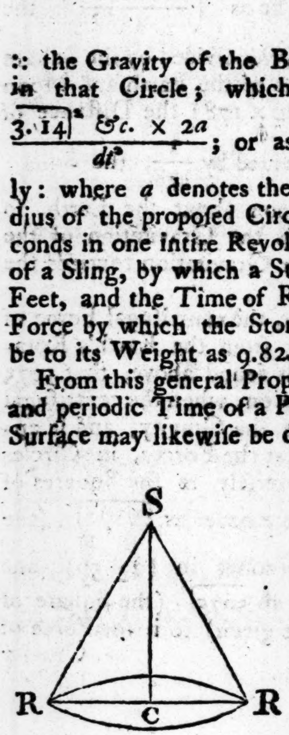
In like ſort the Ratio of the Forces of Gravitation of the Moon, towards the Sun and Earth, may be computed. For, the centrifugal Forces in Circles, being univerſally as the Radii apply'd to the Squares of the Times of Revolution, it will be as $\left(\frac{81000000}{1}\right)$ the Semi-diameter of the *Magnus Orbis* divided by the Square of one Year (the periodic Time of the Earth and Moon about the Sun) is to (240000×178) the Diſtance of the Moon from the Earth divided by $\frac{1}{178}$, the Square of the periodic Time of the Moon about the Earth, ſo is 1,9 to 1 nearly; and ſo is the Gravitation of the Moon towards the Sun to her Gravitation towards the Earth.

Alſo, after the ſame Manner, the centrifugal Force of a Body at the Equator, ariſing from the Earth's Rotation, is derived. For ſince it is found above, that 5075 Seconds is the Time of Revolution, when the centrifugal Force would become equal to the Gravity, and it appears *(by Caſe* 2. *Corol.* 2.*)* that the Forces, in Circles having the ſame Radii, are inverſely as the Squares of the periodic Times, we therefore have, as $\overline{86160}|^2$ (the Square of the Number of Seconds in $(\overset{H}{23}\ \overset{M}{56})$ one whole Rotation of the Earth) to $\overline{5075}|^2$ (the Square of the Number of Seconds above given) ſo is the Force of

Gravity (which we will denote by Unity) to $\frac{1}{289}$, the centrifugal Force of a Body at the Equator ariſing from the Earth's Rotation.

But, to determine, in a more general Manner, the Ratio of the Force of a Body revolving in any given Circle, to its Gravity, we have already given 3.14 &c. $\times \sqrt{\frac{2r}{d}}$ for the Time of Revolution at the Surface of the Earth, when the Gravity and centrifugal Force are equal: Therefore, if the Time of Revolution in any Circle whoſe Radius is a, be denoted by t, it follows, from *Corol.* 2. *laſt Prop.* that, $\frac{r}{\overline{3.14}|^2 \text{ \&c.} \times \frac{2r}{d}} : \frac{a}{t^2}$ :: the Gravity of the Body : to its centrifugal Force in that Circle; which, therefore, is as Unity to $\frac{\overline{3.14}|^2 \text{ \&c.} \times 2a}{dt^2}$; or as 1 to $1.228 \times \frac{a}{t^2}$ very nearly: where a denotes the Number of Feet in the Radius of the propoſed Circle, and t the Number of Seconds in one intire Revolution. So that, if the Length of a Sling, by which a Stone is whirled about, be two Feet, and the Time of Revolution ½ of a Second, the Force by which the Stone endeavours to fly off, will be to its Weight as 9.824 to Unity.

From this general Proportion, the centrifugal Force and periodic Time of a Pendulum deſcribing a conical Surface may likewiſe be deduced.



For let SR, the Length of the Pendulum, be denoted by g; the Altitude CS of the Cone, by c; the Semi-diameter CR of the Baſe by a; and the Time of Revolution by t: Then, the Force of Gravity being re-

repreſented by Unity, the Force with which the revolving Body at R, the End of the Pendulum, tends to recede from the Center C, will be defined by $\frac{\overline{3.14\ \&c.}|^{2} \times 2a}{dt^{2}}$, as has been already ſhewn. Therefore, becauſe the Body is retained in the Circle RR by the Action of three different Powers, *i. e.* the centrifugal Force $\left(\frac{\overline{3.14\ \&c.}|^{2} \times 2a}{dt^{2}}\right)$ in the Direction CR, the Force of Gravity (1) in a Direction parallel to SC, and the Force of the Thread or Wire RS, compounded of the former two; it follows, from the Principles of Mechanics, that as SC (c) to CR (g), ſo is the Weight of the Body at R, to the Force with which it acts upon the Thread or Wire RS; and as $1 : \frac{\overline{3.14\ \&c.}|^{2} \times 2a}{dt^{2}}$ :: CS (c) : CR (a): Whence $dt^{2} = \overline{3.14\ \&c.}|^{2} \times 2c$, and $t = 3.14\ \&c. \times \sqrt{\frac{2c}{d}} = 1{,}108\sqrt{c}$ nearly. Becauſe dt^{2}, or its Equal $\overline{3.14\ \&c}|^{2} \times 2c$, expreſſes the Space a heavy Body will deſcend, by its own Gravity, in the Time t *, and ſince $1^{2} : \overline{3.14\ \&c.}|^{2}$ :: $2c$: $\overline{3.14\ \&c}|^{2} \times 2c$ ($= dt^{2}$) it therefore appears that, as the Square of the Diameter of any Circle, is to the Square of its Periphery, ſo is twice the perpendicular Altitude of the Cone, to the Diſtance a heavy Body will freely deſcend in the Time of one whole Gyration of the Pendulum, let the Baſe of the Cone and the Length of the Pendulum be what they will.

PROPOSITION VII.

218. *To determine the Ratio of the Velocities of Bodies deſcending, or aſcending, in Right-lines, when accelerated, or retarded, by Forces, varying according to a given Law.*

Suppoſe a Body to move in the Right-line CH, and let the Force whereby it is urged towards C, or H,

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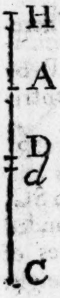
* Art. 202.

be as any variable Quantity F: Moreover, let the Velocity of the Body be repreſented by v; putting its Diſtance CD, from the Point $C = x$, and $Dd = \dot{x}$.



Then, ſince the Time wherein the Space Dd $(\dot{x})$ would be uniformly deſcribed, with the Velocity at D, is known to be as $\frac{\dot{x}}{v}$, the Velocity that would be uniformly generated, or deſtroyed, in that Time by the Force F (being as the Time and Force conjunctly) will conſequently be as $\frac{F\dot{x}}{v}$: Which therefore muſt be equal to, $\pm \dot{v}$, the uniform Increaſe or Decreaſe of Celerity in that Time; and conſequently $\pm v\dot{v} = F\dot{x}$. From whence, when the Value of F is given in Terms of x, or v, the Value of v will likewiſe be known. *Q. E. I.*

COROLLARY I.

219. Hence, the Law of the Velocity being given, that of the Force is deduced: For, ſince $F\dot{x} = \pm v\dot{v}$, it is evident that $F = \pm \frac{v\dot{v}}{\dot{x}}$.

COROLLARY II.

220. Hence, alſo, the Ratio of the Velocity at D to that whereby a Body might revolve in the Periphery of a Circle about the Center C, at the Diſtance of CD, will be known: For, if this laſt Velocity be denoted by w, the Value of F will be rightly expreſſed by $\frac{w^2}{x}$ *: Whence, by Subſtitution, we have $\pm v\dot{v} = \frac{w^2\dot{x}}{x}$, or

* Art. 212.

$\pm v^2 \times \frac{\dot{v}}{v} = w^2 \times \frac{\dot{x}}{x}$: Whence $w^2 : v^2 : \pm \frac{\dot{v}}{v} : \frac{\dot{x}}{x}$, and conſequently $w : v :: \sqrt{\pm \frac{\dot{v}}{v}} : \sqrt{\frac{\dot{x}}{x}}$. Where, as well as above, the Sign of $\dot{v}$ muſt be taken + or — according as the Body is urged from, or towards the Center C.

PROPOSITION VIII.

221. *Suppoſing a Body, let go from a given Point* A *with a given Celerity (c) along a Right-line* CH, *to be urged, either way, in that Line, by a Force varying as any Power (n) of the Diſtance from a given Point* C; *to find, not only, the Relation of the Velocities, and Spaces gone over, but alſo the Times of Aſcent and Deſcent.*

The Conſtruction of the preceding Problem being retained, F will here be expounded by x^n, and we ſhall therefore have $\pm v\dot{v}$ $(= F\dot{x}) = x^n \dot{x}$; and conſequently, by taking the Fluent thereof, $\pm \frac{vv}{2} = \frac{x^{n+1}}{n+1}$; but to correct the Fluent thus found, let x be taken = CA (which we will call a) then v being $= c$, the Fluent in that Circumſtance will become $\pm \frac{c^2}{2} = \frac{a^{n+1}}{n+1}$: Therefore the Fluent duly corrected is $\pm \frac{v^2}{2} \mp \frac{c^2}{2} = \frac{x^{n+1} - a^{n+1}}{n+1}$*, or $v^2 \backsim c^2 = \frac{2x^{n+1} \backsim 2a^{n+1}}{n+1}$: Whence v will come out $= \sqrt{c^2 + \frac{\mp 2a^{n+1} \pm 2x^{n+1}}{n+1}}$: Where the Signs of v and x^{n+1} muſt be alike, when both Quantities increaſe, or decreaſe, at the ſame time; that is, when

* Art. 78.

* Art. 220. when the Force, from C, is a repulſive one*; but, unlike, when one increaſes while the other decreaſes, or the Force, tending to C, is an attractive one. In the former Caſe we therefore have $v = \sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}$; and, in the latter, $v = \sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$.

The Value of v being thus obtained, let the required Time of moving over the Space AD be now denoted by T; then, ſince $\dot{T}$ is univerſally $= \frac{\dot{x}}{v}$, we have $\dot{T}$

$$= \frac{\dot{x}}{\sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}}, \text{ or } \dot{T} =$$

$$\frac{\dot{x}}{\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}}$$ according to the two foreſaid Caſes reſpectively: From whence, by finding the Fluent, the Time itſelf will be known. *Q. E. I.*

COROLLARY.

222. If the Body proceeds from Reſt at A, c will be $= 0$, and we ſhall have $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2x^{n+1} - 2a^{n+1}}}$, or

$$\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2a^{n+1} - 2x^{n+1}}}.$$

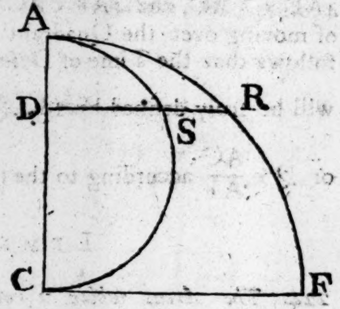
SCHOLIUM.

223. Although, the Fluents of the Expreſſions given above cannot be exhibited, in a general Manner, neither, in finite Terms, nor by means of circular Arcs and Logarithms; yet, in ſome of the moſt uſeful

Caſes that occur in Nature, they may be obtained with great Facility.

Thus, if in $\frac{\overline{1+n}|^{\frac{1}{2}}\dot{x}}{\sqrt{2a^{n+1}-2x^{n+1}}}$ (expreſſing the Fluxion of the Time of Deſcent along AD) n be expounded by 1, 0, $-$ 2, and $-$ 3 ſucceſſively, the Fluxion itſelf will become equal to $\frac{\dot{x}}{\sqrt{a^2-x^2}}$, $\frac{\dot{x}}{\sqrt{2a-2x}}$, $\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, and $\frac{ax\dot{x}}{\sqrt{a^2-x^2}}$ reſpectively: Whence, if ARF be a Quadrant of a Circle whoſe Center is C, and ASC a Semi-circle whoſe Diameter is AG, and DSR be perpendicular to AC; then it will appear,

1°. That, when $n=1$, and $\dot{T}=\frac{\dot{x}}{\sqrt{a^2-x^2}}$, the Velocity ($\sqrt{a^2-x^2}$) at D will be repreſented by DR, and the Fluent ſought by $\frac{AR}{AC}$*.



2°. That, when $n=0$, and $\dot{T}=\frac{\dot{x}}{\sqrt{2a-2x}}$, the Velocity at D, and the Time of Deſcent thro' AD, will each be defined by $\sqrt{2AD}$.

3°. That, when $n=-2$, and $\dot{T}=\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, the Velocity $\left(\frac{\sqrt{ax-xx}}{x\sqrt{\frac{1}{2}a}}\right)$ will be as $\frac{DS}{CD\sqrt{\frac{1}{2}AC}}$, and the Time of Deſcent thro' AD, as $\sqrt{\frac{1}{2}AC}\times\overline{AS+DS}$.

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* Art. 142.

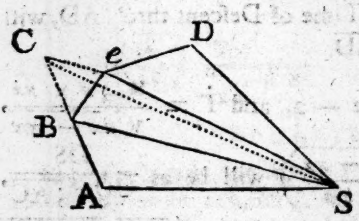
4°. And that, when $n = -3$, and $\dot{T} = \frac{a x \dot{x}}{\sqrt{a^2 - x^2}}$, the Velocity will be as $\frac{DR}{AC \times CD}$, and the Time as $AC \times DR$.

Hence the Time of the whole Deſcent thro' the Radius AC, appears to be as $\frac{AF}{AC}$, $\sqrt{2AC}$, $\sqrt{\frac{1}{2}AC} \times AF$, or AC^2. But the Time of one whole Revolution in

* Art. 215. the Periphery ARF *&c.* was found to be as $\frac{4AF}{AC^{\frac{n+1}{2}}}$ *; which in the four Caſes above ſpecified is $\frac{4AF}{AC}$, $\frac{4AF}{\sqrt{AC}}$, $4AF \times \sqrt{AC}$, and $4AF \times AC$: Therefore, if the Time of moving over the Quadrant AF be denoted by $\mathcal{Q}$, it follows that the Time of Deſcent thro' the Radius AC, will be truly defined by $\mathcal{Q}$, $\mathcal{Q} \times \frac{AC\sqrt{2}}{AF}$, $\mathcal{Q} \times \sqrt{\frac{1}{2}}$, or $\mathcal{Q} \times \frac{AC}{AF}$ according to the foreſaid Caſes reſpectively.

LEMMA.

224. *The Areas which a revolving Body deſcribes, by Rays drawn to the Center of Force, are proportional to the Times of their Deſcription.*



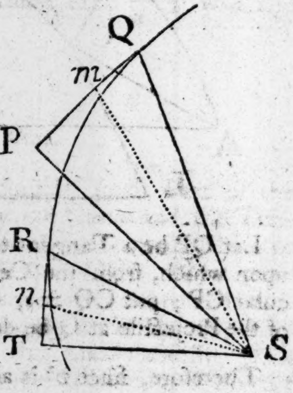
For, let a Body, in any given Time, deſcribe the Right-line AB, with an uninterrupted uniform Motion; but upon its Arrival at B let it be impelled towards the Center S, ſo that, inſtead of proceeding along ABC, it may, after the Impulſe, deſcribe the Right-line B*c*.

Becauſe the Force, acting in the Line SB, can neither add to, nor take from, the Celerity which the Body has in a Direction perpendicular to that Line, the Diſtance of the Body from the ſaid Line, at the end of a given Time, will therefore be the very ſame as if no Force had acted; and conſequently the Area B*c*S equal to the Area BCS, which would have been deſcribed in the ſame time, had the Body proceeded uniformly along BC; becauſe Triangles, having the ſame Baſe and Altitude, are equal.

Therefore ſeeing no Impulſe, however great, can affect the Quantity of the Area deſcribed about the Center S, in a given Time, and becauſe the Areas ASB, BSC, deſcribed about that Point, when no Force acts, are as the Baſes AB, BC, or the Times of their Deſcription, the Propoſition is manifeſt.

COROLLARY.

225. Hence the Velocity of a revolving Body, at any Point Q or R, is inverſely as the Perpendicular SP or ST, falling from the Center of Force upon the Tangent at that Point.

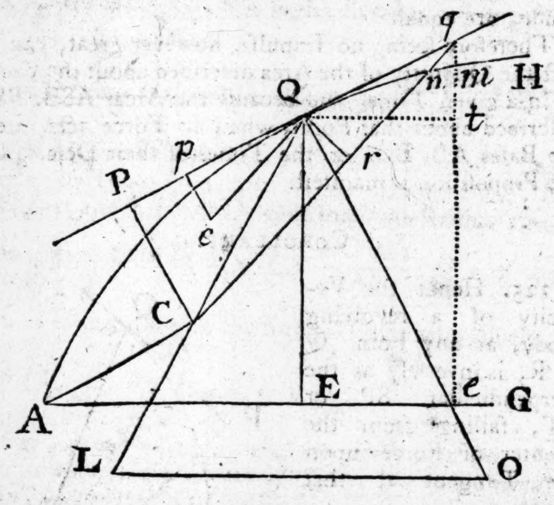


For, let two other Bodies *m* and *n* be ſuppoſed to move uniformly from Q and R, along the Tangents QP and RT, with Velocities reſpectively equal to thoſe of the revolving Body at Q and R; then the Diſtances Q*m* and R*n*, gone over in the ſame Time, will be to each other as thoſe Velocities; and the Areas QS*m* and RS*n* will be equal, being equal

to thoſe deſcribed by the revolving Body in the ſame
* Art.113. time*: Whence $Qm \times SP$ being $= Rn \times ST$, it follows that $Qm : Rn :: ST : SP :: \frac{1}{SP} : \frac{1}{ST}$.

PROPOSITION IX.

226. *To determine the Law of the centripetal Force, tending to a given Point* C, *whereby a Body may deſcribe a given Curve* AQH.



Let QP be a Tangent to the Curve at any Point Q; upon which, from the Center C, let fall the Perpendicular CP; put $CQ = s$, $CP = u$; and let the Velocity of the Projectile at Q be denoted by v.

Therefore, ſince v^2 is always as $\frac{1}{u^2}$ (by the *Corol.* to *Lemma)* it is evident, by taking the Fluxions of both Quantities, that $v\dot{v}$ will alſo be as $\frac{-\dot{u}}{u^3}$: But the centripetal Force, whether the Body moves in a Right-line or a Curve, is always as $-\frac{v\dot{v}}{\dot{s}}$ (by *Art.* 219. and 206.) Therefore the centripetal Force is likewiſe as $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

The ſame otherwiſe.

227. Let the Ray of Curvature QO be denoted by R: Then, becauſe the centripetal Forces in Circles are known to be as the Squares of the Velocities directly and the Radii inverſely *, it follows that the Force, tending to the Point O, whereby the Body might be retained in its Orbit at Q, or in the Circle whoſe Radius is QO, will be defined by $\frac{1}{u^2}\times\frac{1}{R}$: Whence (by the Reſolution of Forces) it will be CP (u) : CQ (s) :: $\frac{1}{u^2R}$ (the Force in the Direction QO) : $\frac{s}{u^3R}$, the Force in the Direction QC: Which, becauſe $R = \frac{s\dot{s}}{\dot{u}}$ † will alſo be expreſſed by $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

* Art. 212.

† Art. 73.

Another Way.

228. Let nq be the indefinitely ſmall Part of the Right-line Cq, intercepted by the Curve and the Tangent Qq, expreſſing the Effect of the centripetal Force in the Time of deſcribing the Area QCn. Now theſe Effects, or the Diſtances deſcended by means of Forces uniformly continued, are known to be in the duplicate Ratio of the Times ‡, or of the Areas denoting thoſe Times §: Therefore, the centripetal Force at Q, or the Diſtance deſcended by means thereof in a given Time, will be as nq applied to the ſecond Power of the Area QCq, or as $\frac{nq}{CP^2\times Qq^2}$. This Expreſſion is the ſame

‡ Art. 201.

§ Art. 224.

with that given by Sir *Iſaac Newton*, in his *Principia*, *Book* 1. *Prop*. 6. But, to adapt it to a fluxional *Calculus*; let QE be an Ordinate to the principal Axis AG; and let (as uſual) AE $= x$, EQ $= y$, AQ $= z$, E*e* (or Q*t*) $= \dot{x}$, Q*q* $= \dot{z}$; ſuppoſing *eq* (parallel to EQ) to interſect the Curve and the Tangent in *m* and *q*.

Since Q*q* is conceived indefinitely ſmall (or in its naſcent State) the Triangle *nmq* may be taken as rectilineal *; alſo the Angle $n =$ CQP and the Angle $m =$ Q*qt*: Whence, it will be (by Trigonometry) as *S*. CQP (n) : *S*. Q*qt* (m) :: $mq : nq$; that is, as $\frac{CP}{CQ} : \frac{Qt}{Qq}$ :: $mq : nq = \frac{CQ \times Qt \times mq}{CP \times Qq}$: Which ſubſtituted above gives $\frac{CQ \times Qt \times mq}{CP^3 \times \overline{Qq}^3}$ for the Meaſure of the centripetal Force at Q: But *mq* (ſuppoſing *x* to flow uniformly) is known to be as $-\ddot{y}$: Therefore the Force at Q, is as $\frac{CQ \times Qt \times -\ddot{y}}{CP^3 \times Qq^3}$, or its Equal $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$; where the Diviſor $(u^3\dot{z}^3)$ is as the Cube of (QC*q*) the Fluxion of the Area AQC.

* Art. 136.

The very ſame Theorem may likewiſe be deduced from that given by our ſecond Method: For, ſince (R) the Ray of Curvature at Q is univerſally * $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, the Value of $\frac{s}{u^3 R}$ (there found) will here, by Subſtitution, become $= \frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$.

† Art.68.

This Expreſſion, tho' in appearance leſs ſimple than $\frac{\dot{u}}{u^3\dot{s}}$, firſt found, is, for the general part, more commodious in Practice.

Co-

COROLLARY I.

229. If the Point C be ſo remote that all Right-lines drawn from thence to the Curve may be conſider'd as parallel to each other, the Force will then (making Qr perpendicular to Cq) be as $\frac{-s\dot{x}\ddot{y}}{Cq\times\overline{Qr}^3}$, or barely as $\frac{-\dot{x}\ddot{y}}{\overline{Qr}^3}$; ſince s (Cq) in this Caſe may be rejected.

From this Expreſſion, which is general, in all Caſes where the Force acts in the Direction of parallel Lines, it appears that the Force, which always acting in the Direction of the Ordinate QE, would retain the Body in its Orbit, is every where as $\frac{-\ddot{y}}{\dot{x}^2}$; becauſe QC here coincides with QE, and Qr becomes $=\dot{x}$.

COROLLARY II.

230. Becauſe the Force, tending to the Point C, is univerſally as $\frac{CQ}{CP^3\times QO}$ $\left(\text{or } \frac{s}{u^3R}\right)$ the Force to any other Point c, will, by the ſame Argument, be as $\frac{cQ}{\overline{cp}^3\times QO}$. Hence the Forces, to different Centers C and c (about which equal Areas are deſcribed in the ſame time) are to each other in the Ratio of $\frac{CP^3}{CQ}$ to $\frac{\overline{cp}^3}{CQ}$ inverſely.

COROLLARY III.

231. Moreover, the Ratio of the Velocity at Q to the Velocity whereby the Body might revolve in a Circle about the Center at C, at the Diſtance CQ, is eaſily deduced from hence: For, ſince the Celerity at Q is that

whereby the Body might revolve in a Circle about the Center O, and the Forces tending to the Centers O and C are to each other as u (CP) and s (CQ); it therefore follows, if the Ratio ſought be aſſumed as v to w, that $\frac{v^2}{QO} : \frac{w^2}{QC} :: u : s$ (*by Art.* 212.) Whence alſo $v^2 : w^2 :: u \times QO\ (uR) : s \times QC\ (s^2)$ and conſequently $v : w :: \sqrt{\frac{uR}{ss}} : 1 :: \sqrt{\frac{u\dot{s}}{\dot{u}\,s}} : 1 :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{\frac{\dot{u}}{u}}$ (becauſe $R = \frac{s\dot{s}}{\dot{u}}$).

The ſame Proportion may alſo be derived from *Corol.* 2. *Prop.* 7. For it is there proved that $v : w :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{-\frac{\dot{v}}{v}}$; and it appears from above, that $-\frac{\dot{v}}{v} = \frac{\dot{u}}{u}$: Whence the whole is manifeſt.

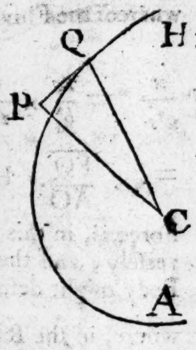
If OL be made perpendicular to QC, QL will be $\left(= \frac{CP \times QO}{CQ}\right) = \frac{uR}{s}$, and $\frac{QL}{CQ} = \frac{uR}{s^2}$; and therefore $v : w :: QL^{\frac{1}{2}} : CQ^{\frac{1}{2}}$: Which is another Proportion of the propoſed Celerities.

COROLLARY IV.

232. Laſtly, the Law of centripetal Force being given, the Nature of the Trajectory AQ may from hence be found; for ſince the Force (F) is univerſally defined by $\frac{\dot{u}}{u^3 \dot{s}}$, it is evident that $\frac{-1}{2u^2}$ will be = the Fluent of $F\dot{s}$; which, when F is given in Terms of s, will become known; and then, the Relation between u and s being given, the Curve itſelf is known.

EXAMPLE I.

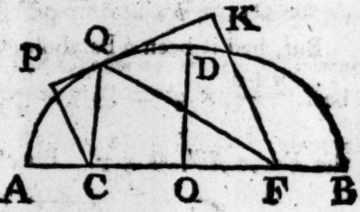
233. *Let the given Curve* AQH *be the logarithmic Spiral, and* C *the Center thereof:* Then u (CP) being in this Caſe $= \frac{b\dot{s}}{a}$*, we have $\frac{\dot{u}}{u^3\dot{s}}$† $\left(= \frac{b\dot{s}}{a\dot{s}} \times \frac{a^3}{b^3s^3}\right)$ $= \frac{a^2}{b^2s^3}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$‡ $\left(= \sqrt{\frac{bs\dot{s}}{a} \times \frac{a}{bs\dot{s}}}\right) =$ Unity. Hence it appears that the Force is inverſely as the Cube of the Diſtance; and the Velocity, every where, equal to that whereby the Body might revolve in a Circle at the ſame Diſtance.



* Art. 61.
† Art. 227.
‡ Art. 231.

EXAMPLE II.

234. *Let it be required to find the Law of the centripetal Force, whereby a Body, tending to the Focus* C, *is made to revolve in the Periphery of an Ellipſis* AQDB.



From the other Focus F draw FK parallel to CP meeting the Tangent PQ (at Right-angles) in K, join F, Q; putting the tranſverſe Axis AB $= a$, the Semi-conjugate OD $= \frac{1}{2}b$, and the Parameter $\left(\frac{b^2}{a}\right) = p$: Then, CQ and CP being denoted as above*, we have FQ ($=$AB$-$CQ) $= a - s$; whence, by reaſon of the ſimilar Triangles CQP and FQK, it will be

* Art. 231.

$s : u :: a - s : \mathrm{FK} = \frac{\overline{a-s} \times u}{s}$. But FK × CP is $= \mathrm{OD}^2$ (by the Nature of the Curve.) Hence we get $\frac{\overline{a-s} \times u^2}{s} = \frac{1}{4} b^2$; and conſequently $\frac{1}{u^2} = \frac{4a}{b^2 s} - \frac{4}{b^2}$; whereof the Fluxion being $-\frac{2\dot{u}}{u^3} = -\frac{4a\dot{s}}{b^2 s^2}$, we obtain

* Art. 227. † Art. 231. $\frac{\dot{u}}{u^3 \dot{s}}$ * $= \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ † $= \sqrt{\frac{2 \times \overline{a-s}}{a}}$ $= \sqrt{\frac{\mathrm{FQ}}{\mathrm{AO}}}$. Hence, it appears that the centripetal Force is, in this Caſe, as the Square of the Diſtance inverſely; and the Velocity at Q is to that whereby the Body might deſcribe a Circle at the Diſtance CQ, every where, in the Ratio of $\mathrm{FQ}^{\frac{1}{2}}$ to $\mathrm{AO}^{\frac{1}{2}}$.

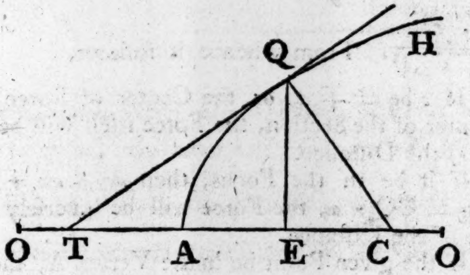
If the Curve had been an Hyperbola; then $\frac{a+s}{s} \times a^2$ (inſtead of $\frac{a-s}{s} \times u^2$) would have been $= \frac{1}{4} b^2$; and ſo $\frac{\dot{u}}{u^3 \dot{s}} = \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, the very ſame as before,

But, had it been a Parabola, the Equation would have been $\frac{a+0}{s} \times u^2 = \frac{1}{4} b^2$, or $\frac{u^2}{s} \left(= \frac{b^2}{4a}\right) = \frac{1}{4} p$; and the Force *ſtill*, as $\frac{2}{ps^2}$. But, the Meaſure of the Velocity $\left(\sqrt{\frac{u\dot{s}}{s\dot{u}}} = \sqrt{\frac{2a-2s}{a}}\right)$ in this Caſe becoming barely $= \sqrt{2}$, it follows that the Velocity in a Parabola is to that whereby the Body might deſcribe a Circle at the ſame Diſtance from the Center, in the conſtant Ratio of $\sqrt{2}$ to Unity.

E X-

EXAMPLE III.

235. *Let it be required to find the Law of the centripetal Force, by which a Body, tending to any given Point* C, *in the Axis, is made to deſcribe a conic Section* AQH.



Put the ſemi-tranſverſe Axis (OA) $= a$, the ſemi-conjugate $= b$, and the given Diſtance of the Point C from the Vertex A $= c$: Put alſo the Abſciſſa AE, $= x$, the Ordinate EQ $= y$, and CQ $= s$ *(as before)*.

The Area of the Triangle ECQ being ($= \frac{1}{2}$EC×EQ) $= \frac{cy - xy}{2}$, its Fluxion is therefore $= \frac{c\dot{y} - x\dot{y} - y\dot{x}}{2}$; which added to $y\dot{x}$, the Fluxion of the Area AEQ, gives $\frac{c\dot{y} + y\dot{x} - x\dot{y}}{2}$ for the Fluxion of the whole Area ACQ deſcribed about the Center of Force. Whence (*by Art.* 228.) the required centripetal Force at Q will be as $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3}$. Which Expreſſion is general, let the Curve be of what Kind it will. But in the Caſe above, y being $= \frac{b}{a}\sqrt{2ax \pm x^2}$, we have $\dot{y} = \frac{b\dot{x} \times \overline{a \pm x}}{a\sqrt{2ax \pm x^2}}$, $\ddot{y} = \frac{-ab\dot{x}^2}{\overline{2ax \pm x^2}|^{\frac{3}{2}}}$, and $c\dot{y} + y\dot{x} - x\dot{y} =$

$\frac{b\dot{x} \times \overline{ca + ax \pm cx}}{a\sqrt{2ax \pm x^2}}$; and therefore, by ſubſtituting theſe Values, we get $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3} = \frac{a^4 s}{b^2 \times \overline{ca + ax \pm cx}|^3}$ Which, becauſe $\frac{a^4}{b^2}$ is conſtant, will alſo be as $\frac{s}{\overline{ca + ax \pm cx}|^3}$. From whence it follows,

1°. If c be $= \mp a$, or the Center of Force be in the Center of the Section, the Force itſelf will be barely as $(\pm s)$ the Diſtance.

2°. If it be in the Focus, then $ac + ax \pm cx$ becoming $= CQ \times a$, the Force will be inverſely as the Square of the Diſtance.

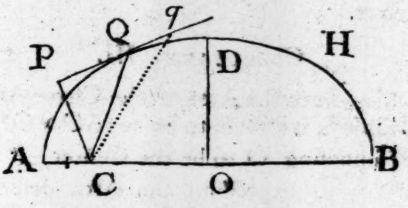
3°. If the given Point be in the Vertex A, the Force will be as $\frac{s}{x^3}$: Which therefore in the Circle (where $x = \frac{s^2}{2a}$) will be as $\frac{1}{s^5}$, or the fifth Power of the Diſtance reciprocally.

4°. Laſtly, if the Point C be at an indefinite Diſtance from the Vertex, or the Force be ſuppoſed to act in the Direction of Lines parallel to the Axis AO; then the Force will be as the Cube of OE inverſely.

PROPOSITION X.

236. *To determine the Ratio of the Velocities of Bodies revolving in different Orbits, about the ſame, or different, Centers; the Orbits themſelves, and the Forces whereby they are deſcribed, being given.*

Let AQH be any Orbit, deſcribed about the Center of Force C, and let the Force itſelf at the principal Vertex A be denoted by F; alſo let r ſtand for the Semiparameter, or the Ray of Curvature at the Vertex, and let CP be perpendicular to the Tangent QP.



Then, the Celerity at A being, always, as $\sqrt{rF}$ (*by Art.* 212.) we have CP : CA :: $\sqrt{rF}$ (the Velocity at A) to $\frac{CA \times \sqrt{rF}}{CP}$, the Velocity at Q (*by Art.* 225.) Which anſwers in all Caſes, let the Values of AC, r and F be what they will. *Q. E. I.*

Corollary I.

237. If the centripetal Force be as the Square of the Diſtance inverſely, or F be expounded by $\frac{1}{AC^2}$, the Velocity at Q will become $\frac{AC}{CP} \times \sqrt{\frac{r}{AC^2}}$, or $\frac{\sqrt{r}}{CP}$: Whence the Velocities, in different Orbits, about the ſame Center, are in the ſubduplicate Ratio of the Parameters, and the inverſe Ratio of the Perpendiculars from the Center of Force to the Tangents, conjunctly.

Corollary II.

238. Hence, if the Celerity at Q be denoted by Qq, and Cq be drawn; then, Qq being as $\frac{\sqrt{r}}{CP}$, it follows that $\sqrt{r}$ is as CP × Qq, or as the Triangle QCq: There-

fore the Areas deſcribed about a common Center of Force in a given Time, are in the ſubduplicate Ratio of the Parameters.

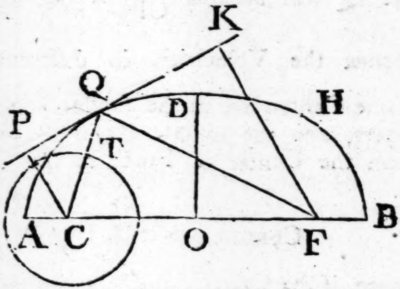
COROLLARY III.

239. Laſtly, ſince the Area of the Curve AQHB *&c.* when an Ellipſe*, is known to be as $(AO \times OD)$ $AO \times \sqrt{r \times AO}$ (ſuppoſing O to be the Center) if the ſame be apply'd to $\sqrt{r}$, expreſſing the Area deſcribed in a given Part of Time (by the laſt *Corol.*) we ſhall thence have $AO \times \sqrt{AO}$, or $AO^{\frac{3}{2}}$ for the Meaſure of the Time of one whole Revolution. From whence it appears, that the periodic Times, let the Species of the Ellipſes be what they will, are in the ſeſquiplicate Ratio of their principal Axes.

* Art. 234.

PROPOSITION XI.

240. *The centripetal Force, tending to a given Point* C, *being as the Square of the Diſtances reciprocally, and the Direction and Velocity of a Body at any Point* Q *being given; to determine the Path in which the Body moves, and the periodic Time, in caſe it returns.*



It is evident from *Art.* 234. *and* 235. that the Trajectory AQB is a conic Section; whereof the Point C is one of the *Foci.*

Let F be the other Focus, and upon the Tangent PQK let fall the Perpendiculars CP and FK, and let CQ and FQ be drawn: Alſo put the ſemi-tranſverſe Axis $AO = a$, the given focal Diſtance $CQ = d$, and the Sine of the Angle of Direction CQP (to the Radius 1) $= m$; and let the given Velocity at Q be to that whereby the Body might revolve in a Circle about the Center C, at that Diſtance, in any given Ratio of n to Unity: Then it will be $n : 1 :: FQ^{\frac{1}{2}} : AO^{\frac{1}{2}}$ (by *Art.* 234.) therefore $n^2 : 1^2 : FQ\ (2a-d) : AO\ (a)$; whence AO ($a$) is given $= \frac{d}{2-n^2}$. Moreover, ſince $CP = m \times CQ$, and $FK = m \times FQ$, we have OD^2 ($= CP \times FK = m^2 \times CQ \times FQ = \frac{m^2n^2d^2}{2-n^2}$; whence the ſemi-conjugate Axis (OD) is given likewiſe.

Laſtly, it will be (by *Art.* 239.) as $CT^{\frac{3}{2}} : AO^{\frac{3}{2}} ::$ (*P*) the periodic Time in any given Circle, whoſe Radius is CT, to $\left(\frac{AO^{\frac{3}{2}}}{CT^{\frac{3}{2}}} \times P\right)$ the required Time of one Revolution when the Orbit is an Ellipſis; that is, when n^2 is leſs than 2: For, if n^2 be $= 2$, the Curve (as its Axis $\frac{2d}{2-n^2}$ becomes infinite) will degenerate to a Parabola; and, if n^2 be greater than 2, the Axis being negative, it is then an Hyperbola; whoſe two principal Diameters are equal to $\frac{2d}{n^2-2}$ and $\frac{2mnd}{\sqrt{n^2-2}}$. *Q. E. I.*

COROLLARY.

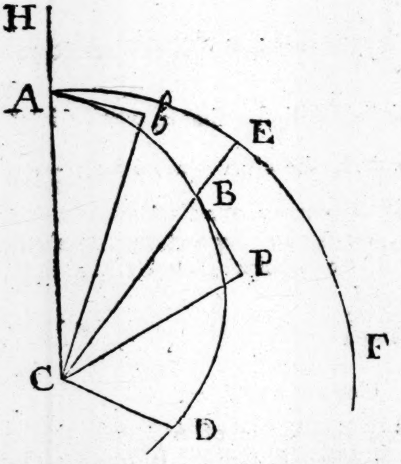
241. Seeing neither the Value of AO, nor that of the periodic Time, is affected with m, it follows that the principal Axis, and the periodic Time, will remain in-

invariable, if the Velocity at Q be the ſame, let the Direction at that Point be what it will.

The ſame Solution may likewiſe be brought out, from *Art.* 238. by firſt finding the *principal Parameter:* For, it is evident that the Area deſcribed by the Body about the Center C, in any given Time, is to the Area deſcribed, in the ſame Time, by another Body revolving in a Circle at the Diſtance CQ, as *mn* to Unity: Hence, it will be $1^2 : m^2n^2 :: d : (m^2n^2d)$ the Semi-parameter*: From which (proceeding as above) we get $a \times m^2n^2d$ $(=OD^2) = m^2 \times \overline{2ad - d^2}$; and conſequently $a = \frac{d}{2 - n^2}$, the ſame as before.

PROPOSITION XII.

242. *The centripetal Force being as any Power (n) of the Diſtance, and the Direction and Velocity of a Body at any Point* A *being given, to determine the Orbit or Trajectory.*



From the Center of Force C, to any Point B in the required Trajectory ABD, let CB be drawn; join C, A, and let A*b* be the given Direction of the Body at the Point A, and C*b* perpendicular thereto; alſo let the Velocity at A be to that whereby a Body might deſcribe a Circle AEF, about the Center C, in any given Ratio of *p* to Unity; putting CA=*a*, and CB=*x:* Then becauſe this laſt Velocity (the centripetal Force being as x^n (or a^n) is rightly defined by $a^{\frac{n+1}{2}}$ *, the Velocity of the Body at A will be truly expreſſed by $pa^{\frac{n+1}{2}}$.

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* Art. 238.

Moreover, it is proved in *Art.* 221. *and* 206. that if the Celerity, at any given Diſtance a from the Center, be denoted by c, the Celerity at any other Diſtance x will be truly repreſented by $\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$:

Whence, $pa^{\frac{n+1}{2}}$ being ſubſtituted for c, we have $\sqrt{\overline{p^2 + \frac{2}{n+1}} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}$ for the Celerity at B.

But now, to determine the Curve itſelf from hence, let BP be a Tangent to it at B, and CP perpendicular to BP; alſo let CB, produced, meet the Periphery of the Circle in E; putting the Arch AE $=z$, the foreſaid Velocity at B (to ſhorten the Operation) $=v$, and C$b=b$: Then it will be (by *Art.* 225.) $v : c$ (the Velocity at A) :: b : CP$= \frac{bc}{v}$ Whence BP ($=\sqrt{CB^2 - CP^2}$) $= \frac{\sqrt{x^2v^2 - b^2c^2}}{v}$.

Moreover (by *Art.* 35.) we have, as CB : CP :: v : $\left(\frac{CP}{CB} \times v\right)$ the Velocity of the Body at B in a Direction perpendicular to CE; and conſequently, as CB : CE :: $\frac{CP}{CB} \times v$ (the ſaid Velocity) to $\frac{CP \times CE}{CB^2} \times v$ the angular Velocity of the Point E (revolving with the Body.) By the ſame *Article*, the Velocity at B in the Di-

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* Art. 214.

Direction CBE will be $\frac{BP}{CB}\times v$: Therefore, the Velocity of E being to the Velocity of B, in the ſaid Direction, as $\frac{CP\times CE}{CB^2}$ to $\frac{BP}{CB}$, the Fluxions of AE (z) and CB (x) muſt conſequently be in that Ratio; that is, $\frac{CP\times CE}{CB^2}:\frac{BP}{CB}::\dot{z}:\dot{x}$; whence $\dot{z}=\frac{CP\times CE}{CB\times BP}\times\dot{x}=\frac{bc}{v}\times\frac{a}{x}\times\frac{v\dot{x}}{\sqrt{x^2v^2-b^2c^2}}=\frac{abc\dot{x}}{x\sqrt{x^2v^2-b^2c^2}}=\frac{ab\dot{x}}{x\sqrt{\frac{x^2v^2}{c^2}-b^2}}$. Which Equation is general, let the Law of the centripetal Force be what it will: But in the Caſe above propoſed, v^2 being $=\overline{p^2+\frac{2}{n+1}}\times a^{n+1}-\frac{2x^{n+1}}{n+1}$, and $c^2=p^2a^{n+1}$; it becomes $\dot{z}=\frac{abp\dot{x}}{x\sqrt{\overline{p^2+\frac{2}{n+1}}\times x^2-p^2b^2-\frac{2x^{n+3}}{\overline{n+1}\times a^{n+1}}}}$; whoſe Fluent is the Meaſure of the angular Motion; from which, when found, the Orbit may be conſtructed: Becauſe, when AE, or the Angle ACE is given, as well as CB, the Poſition of the Point B is alſo given. But this Value of $\dot{z}$ is indeed too complex to admit of a Fluent in algebraic Terms, or even by circular Arcs and Logarithms, except in certain particular Caſes; as when the Exponent n is equal to 1, — 2, — 3, or — 5; beſides ſome others wherein the Values of p and n are related in a particular Manner. *Q. E. I.*

Co-

COROLLARY I.

243. If the given Velocity at A be ſuch that $p^2 + \frac{2}{n+1} = 0$, or $p = \sqrt{\frac{-2}{n+1}}$ (which is always poſſible when the Value of $n+1$ is negative) our Equation will become $\dot{z} \times \frac{abp\dot{x}}{x\sqrt{-p^2b^2 + \frac{p^2x^{n+3}}{a^{n+1}}}}$: Which, by putting $n+3=m$, *&c.* is reduced to $\dot{z} = \frac{ab\dot{x}}{x\sqrt{-b^2+\frac{x^m}{a^{m-2}}}}$: Whereof the Fluent will be found (by the ſecond Part of this Work (equal to $\pm \frac{2a}{m}$ multiply'd by the Difference of the two circular Arcs, whoſe Secants are $\frac{x^{\frac{1}{2}m}}{ba^{\frac{1}{2}m-1}}$ and $\frac{a}{b}$ to the Radius Unity.) From this Value of the Arch AE the Poſition of the Point B, in the Orbit, is given.

But if the Angle of Direction CA*b* be a right one, the Fluent will become barely $= \pm \frac{2a}{m} \times$ Arch whoſe Secant is $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}m}}$ (becauſe then $b=a$, and the Arch whoſe Secant is $\frac{a}{b}, = 0$) which therefore when $x^{\frac{m}{2}}$ becomes in-

infinite, will be truly defined by $\pm\frac{1}{2m}$ × whole Periphery AF, *&c.* Whence it is evident that the Body muſt either fly intirely off, or fall to the Center C, in a Number of Revolutions expreſſed by $\pm\frac{1}{2m}$; according as the Value of m is poſitive or negative.

Thus, if $n = -2$, and $m = 1$, the Body will fly intirely off in half a Revolution: And, if $n = -4$, and $m = -1$, it will fall to the Center in half a Revolution.

COROLLARY II.

244. Moreover, tho' the Fluent expreſſing the Angle at the Center cannot be exhibited in a general Manner yet there are certain Caſes of the Exponent (n) where its reſpective Values may be derived from each other.

For let (as above) $n+3$ be put $= m$, and (to ſhorten the Operation) let CA (a) be taken as Unity: Then our Equation will be transformed to $\dot{z} =$

$$\frac{b\dot{x}}{x\sqrt{\overline{1+\frac{2}{\overline{m-2}\cdot p^2}} \times x^2 - b^2 - \frac{2x^m}{\overline{m-2}\cdot p^2}}}$$: Make

$y = x^{\frac{m}{2}}$, and it will be farther transformed to $\dot{z} =$

$$\frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{\overline{1+\frac{2}{\overline{m-2}\cdot p^2}} \times y^{\frac{4}{m}} - b^2 - \frac{2y^2}{\overline{m-2}\cdot p^2}}}:$$

Put $r = \frac{4}{m}$, and it will become $\dot{z} = \frac{2}{m} \times$

$$\frac{b\dot{y}}{y\sqrt{\frac{ry^2}{\overline{r-2}\cdot p^2} - b^2 + \overline{1-\frac{r}{\overline{r-2}\cdot p^2}} \times y^r}}$$: Laſtly,

2 let

let $\frac{r}{\overline{r-2}.p^2} = 1 + \frac{2}{\overline{r-2}.q^2}$ (or $1 - \frac{r}{\overline{r-2}.p^2} = -\frac{2}{\overline{r-2}.q^2}$, or $q^2 = \frac{2p^2}{r-p^2\times\overline{r-2}}$) and then we ſhall have $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{\overline{1+\frac{2}{\overline{r-2}.q^2}}\times y^2 - b^2 - \frac{2y^r}{\overline{r-2}.q^2}}}$. Which Expreſſion (excepting the general Multiplicator $\frac{2}{m}$) being exactly of the ſame Form with the firſt above given, muſt therefore be the Fluxion of the Angle at the Center, when the Index of the Force is $r-3$; for the very ſame Reaſons that the former appears to be the Fluxion thereof when the Index is $m-3$ (or n.)

Hence, if the Fluent of $\frac{b\dot{y}}{y\sqrt{\overline{1+\frac{2}{\overline{r-2}.q^2}}\times y^2 - b^2 - \frac{2y^r}{\overline{r-2}.q^2}}}$, or the Angle at the Center, when the Exponent is $r-3$ (or $\frac{4}{m} - 3 = \frac{4}{n+3} - 3$) be denoted by w, the Value of z, (the Meaſure of the ſaid Angle, when the Exponent is $m-3$ (or n) will be truly defined by $\frac{2w}{m}$.

From which we collect that, if the Indices of the Force, in any two Caſes, be repreſented by n and $\frac{4}{n+3} - 3$, and the reſpective Diſtances from the Center by x and $x^{\frac{n+3}{2}}$, then the Angles themſelves correſponding to thoſe Diſtances will be every where in the conſtant Ratio of 2 to $n+3$. Therefore, when the Orbit can be conſtructed in the one Caſe, it alſo may in the other, provided the above Equation $q^2 \left(= \frac{2p^2}{r - p^2 \times \overline{r-2}}\right) = \frac{\overline{n+3}.p^2}{2+\overline{n+1}.p^2}$, for the Relation of the Celerities at A, does not become impoſſible, as it will, ſometimes, when n is a negative Number.

COROLLARY III.

245. If the Body be ſuppoſed to move in a vertical Direction AH; then, putting the Velocity $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}} = 0$, we get x (CH) $= \overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}\big|^{\frac{1}{n+1}} \times a$ = the Height to which the Body will aſcend: Hence $\overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}\big|^{\frac{1}{n+1}} \times a - a$ (= AH) is the Diſtance through which it muſt freely deſcend to acquire the given Celerity at A: This Diſtance, in caſe of an uniform Force, when $n = 0$, will become $= \frac{1}{2} p^2 a$: And, when the Force is inverſely as the Square of the Diſtance, it will then be $= \frac{p^2 a}{2 - p^2}$.

But, when $p = 1$, or the Velocity at A is juſt ſufficient to retain a Body in the Circle AEF, AH becomes $= \overline{\frac{3+n}{2}}\big|^{\frac{1}{n+1}} \times a - a$: Which in the two Caſes aforeſaid will be equal to $\frac{1}{2}a$, and a reſpectively; but, infinite, when n is $= -3$.

Co-

COROLLARY IV.

246. When the Value of $n+1$ is poſitive, the Velocity at the Center, where $x=0$, will be barely $=\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; but if the Value of $n+1$ be negative, the Velocity at the Center will be infinite; becauſe, then 0^{n+1} is infinite.

COROLLARY V.

247. Moreover, when $n+1$ is negative and x infinite, the Velocity alſo becomes $=\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; becauſe then $x^{n+1}=0$.

Hence, if the centripetal Force be inverſely as ſome Power of the Diſtance greater than the firſt, the Body may aſcend, *ad infinitum*, and have a Velocity always greater than $\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; which is to, $pa^{\frac{n+1}{2}}$, the given Velocity, at A, as $\sqrt{p^2+\frac{2}{n+1}}$ to p. And this will actually be the Caſe when the Value of $p^2+\frac{2}{n+1}$ is poſitive, or p^2 greater than $\frac{2}{-n-1}$, but not otherwiſe, the ſquare Root of a negative Quantity being impoſſible.

Thus, if $n=-2$, or the Force be inverſely as the Square of the Diſtance, and p^2, at the ſame time, greater than 2 $\left(\frac{2}{-n-1}\right)$ the Body will not only continue to aſcend *in infinitum*, but have a Velocity always greater than that defined by $\sqrt{p^2-2}$, which is its Limit.

Co-

COROLLARY VI.

248. Hence the leaſt Celerity ſufficient to cauſe the Body to aſcend for ever in a Right-line is given. For, putting $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}} = 0$, we have $p = \sqrt{\frac{2}{-n-1}}$. Therefore the leaſt Celerity by which the Body might aſcend for ever, is to that whereby it may revolve in a Circle AEF, as $\sqrt{\frac{2}{-n-1}}$ to Unity. From which it appears that, if the Force be inverſely as any Power of the Diſtance greater than the third, a leſs Velocity will cauſe a Body to aſcend *ad infinitum* than would retain it in a Circle.

SCHOLIUM.

249. From the Ratio of the Velocity $\left(\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}\right)$ wherewith the Body arrives at any Diſtance x from the Center, to that $\left(x^{\frac{n+1}{2}}\right)$ * which it ought to have to revolve in a Circle at the ſame Diſtance, it will not be difficult to determine in what Caſes the Body will be forced to the Center, and in what others it will continue to fly from it *ad infinitum.*

* Art. 214.

For, firſt, if the Angle CAb be acute, or the Body from A begins to deſcend, it will continue to do ſo till it actually arrives at the Center, if the former Velocity, during the Deſcent, be not ſomewhere greater than the latter, or the Quotient $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ greater than Unity; becauſe, if it ever begins to aſcend, it muſt have an *Apſe*, as D (where a Right-line drawn from the Center cuts the Orbit at Right-angles) and there the Celerity muſt evidently be greater than that ſufficient to cauſe the Body to revolve in a Circle.

Secondly, but if the Quantity $\sqrt{p^2+\frac{2}{n+1}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$, in the Acceſs of the Body towards the Center, increaſes ſo as to become greater than Unity, or be every where ſo; then the Velocity at all inferior Diſtances being more than ſufficient to retain a Body in a Circle at any ſuch Diſtance, the Projectile cannot be forced to the Center.

After the ſame Manner, if the Angle CA*b* be obtuſe, or the Body from A begins to aſcend, it will continue to do ſo for ever, when the foreſaid Quantity is always greater than Unity, or, which is the ſame, when the Body, in its Receſs from the Center, has in every Place through which it paſſeth, a Velocity greater than ſufficient to retain it in a Circle at that Diſtance.

It therefore now remains to find in what Laws of the centripetal Force theſe different Caſes obtain: And, firſt, it is eaſy to perceive that when the Value of $n+1$ is poſitive, that of $\sqrt{p^2+\frac{2}{n+1}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$ will, by increaſing x, become equal to nothing. Therefore the Body cannot aſcend for ever in this Caſe: Neither can it deſcend to the Center (except in a Right-line) becauſe the foreſaid Quantity, by diminiſhing x, becomes greater than Unity (or any other aſſignable Magnitude.)

But, if the Value of n be betwixt -1, and -3, the ſaid general Expreſſion, taking x infinite, will alſo become infinite, provided the Value of $p^2+\frac{2}{n+1}$ be poſitive (or p^2 greater than $\frac{2}{-n-1}$). Therefore the

Body, in this Caſe, may aſcend *ad infinitum*, but cannot poſſibly fall to the Center (except in a Right-line) ſince, $\sqrt{-\frac{2}{n+1}}$, the Value of the general Expreſſion, when $x=0$, is greater than Unity.

Laſtly, if *n* be expreſſed by any negative Number greater than -3, or the Law of the Force be inverſely as any Power of the Diſtance greater than the third, the two extreme Values of $\sqrt{p^2+\frac{2}{n+1}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$ will, *ſtill*, be denoted as in the preceeding Caſe; but here the latter of them, $\sqrt{\frac{-2}{n+1}}$, is leſs than Unity. Therefore the Body muſt, in this Caſe, either aſcend for ever, or be forced to the Center; except in one particular Circumſtance, hereafter to be taken notice of.

Now, from theſe Obſervations we gather,

1°. That, when the centripetal Force is as any Power of the Diſtance directly, or leſs than the firſt Power thereof inverſely, the Orbit will always have an higher and a lower *Apſe*; beyond which the Body cannot aſcend or deſcend.

2°. That, when the centripetal Force is inverſely as any Power of the Diſtance (whole or broken) betwixt the firſt and third, the Orbit will alſo have two *Apſides*, if *p* be leſs than $\sqrt{-\frac{2}{n+1}}$; but otherwiſe, only one; in which laſt Caſe the Body, after it has paſſed its Apſe, will continue to recede from the Center *in infinitum*.

3°. That when the Force is inverſely as any Power greater than the third, the Orbit can, at moſt, have but one *Apſe*; but, in ſome Caſes, it will have none at all: And it may be worth while to inquire here, under what Reſtrictions of the Velocity (*p*) this will happen; ſince thereby, beſides being able to know when the Body will be forced to the Center, *&c.* we ſhall fall upon a Circumſtance ſomewhat remarkable and curious.

Now it appears, that, if the Body from A begins to deſcend, it muſt, when it comes to an *Apſe* at D, have a Velocity there greater than is ſufficient to retain it in a Circle; in which Caſe the general Expreſſion

$$\sqrt{\overline{p^2+\frac{2}{n+1}}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$$

(ſo often mention'd above) muſt accordingly be greater than Unity. Let it be therefore made equal to Unity, which is the utmoſt Limit thereof, beyond which the Orbit cannot admit of an *Apſe*; putting at the ſame time $\dot{x}$, or its Diviſor

$$\sqrt{\overline{p^2+\frac{2}{n+1}}\times x^2-p^2b^2-\frac{2x^{n+1}}{\overline{n+1}.a^{n+1}}},$$

in the general Equation of the Orbit, equal to nothing (it being always ſo at the *Apſides.*) Then, from theſe two Equations, duly order'd, we ſhall get $x =$

$$\left.\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\right|^{\frac{1}{n+1}}\times a, \text{ and } p^2 \left(= \frac{x^{n+3}}{a^{n+1}}\right) =$$

$$\left.\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\right|^{\frac{n+3}{n+1}}\times\frac{a^2}{b^2}.$$

Now, it is evident, if the Value of p be greater than is given from the laſt Equation, the Orbit will have an Apſe; but if leſs, it can have none. In the former Caſe, the Body will therefore fly quite off; and in the latter, it will be forced to the Center. But we are now, naturally, led to inquire what will be the Conſequence when the Value of p is neither greater nor leſs, but exactly the ſame as given from the foreſaid Equation: This is the Caſe above hinted at; and here the Body will continue to deſcend for ever in a Spiral, yet never ſo low as to enter within the Circle whoſe Radius CD is $= \left.\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\right|^{\frac{1}{n+1}}\times a$. For, if the contrary were poſſible, the Body, at its Arrival to the Circumference of that Circle, would (becauſe of the foreſaid Equations) not only have a Direction, but alſo Velocity proper to retain it therein; which cannot be, becauſe the Parts of the Orbit on either Side of an Apſe are always ſimilar to each other.

From the ſame Equation, the Value of the Limit will alſo be given when the Angle of Direction CA*b* is obtuſe, or the Body is projected upwards:

For that Equation (as is eaſy to demonſtrate *) admits of two different Roots, or Values of p; the one greater, the other leſs, than Unity: Whereof the former, giving CD (x) leſs than CA, is to be taken in the preceding Caſe, and the latter (making CD greater than CA) in the preſent. And the Body will, either, continue to aſcend for ever, or come to an *Apſis*, and from thence fall to the Center, according as the given Value of p is greater or leſs than that here ſpecified. But if it be neither greater nor leſs, but exactly the ſame, then the Body, tho' it will ſtill continue to aſcend for ever in a Spiral, yet it can never riſe ſo high as the Circumference of the Circle whoſe Radius CD is =

$$\left(\frac{2+\overline{n+1}.p^2}{n+3}\right)^{\frac{1}{n+1}} \times a,$$

for Reaſons ſimilar to thoſe already delivered, in reſpect to the preceding Caſe.

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* *Mathematical Diſſert.* p. 167.

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